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APPLICATION OF DYNAMIC PROGRAMMING TO SELECT TACTICS FOR AIR-TO-GROUND ATTACK UNDER UNCERTAINTY

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Aeronautical Systems Division
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DEPUTY FOR DEVELOPMENT PLANNING
AERONAUTICAL SYSTEMS DIVISION
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FOREWORD

This technical report was prepared by Dr. Clifford D. Fawcett of the Deputy for Development Planning, Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio, and was presented to the Department of Industrial Engineering of the Ohio State University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

This work was directly motivated by experience obtained while employed as a member of the technical staff at Wright-Patterson Air Force Base. In performing effectiveness analyses of various Air Force weapon systems, it has become increasingly apparent that the tactics used in employing the system can be of overriding importance in determining system effectiveness. It is also clear that uncertainty is an unavoidable and crucial factor in decisions relating to future military systems. It is relatively easy to point out and discuss these facts and few rational people will dispute their importance, but it seems to be rare for a system evaluation to include explicit consideration of tactics optimization and uncertainty. This is undoubtedly due to the conceptual and mathematical difficulties that are encountered in so doing, coupled with the practical considerations of limited resources and time available for most weapon system evaluations. With this situation in mind, the work that is described herein was undertaken as a more extensive consideration of these problems than is usually possible.

Publication of this technical documentary report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

ABSTRACT

This work applies dynamic programming and some notions from decision theory. Basic recursive relations are developed for deterministic and Markovian decision processes. Sufficient conditions are stated that assure the optimality of results that these relationships produce. The application deals with the problem of making a rational selection of tactics for air-to-ground attack when faced by uncertainty as to the exact conditions that prevail.

A single aircraft attack on a target is referred to as a "duel." A duel is treated as a multistage decision process with successive aircraft passes at the target corresponding to stages in the decision process. The basic factors to be considered at each stage of the duel are the weapon effectiveness as a function of the number of weapons delivered, the aircraft's ability to survive, and the aircraft's ability to acquire the target and deliver weapons. We seek to determine an optimal policy that indicates the number of weapons to be delivered and the mode of attack to be used at each pass depending on what state of affairs develops as the duel progresses.

The selected policy maximizes the aircraft's return subject to constraints on the number of passes that can be made, the number of weapons available to the aircraft, and the probability of the aircraft surviving the duel. Several different types of return to the attacker are considered. These include the expected value of the number of hits achieved, the probability of at least one hit, and the expected utility of the duel to the attacker where the utility is an arbitrary function of the number of hits achieved.

A principle result from the duel models is an indication of the maximum return as a function of the constraining probability of the aircraft not surviving the duel. This is referred to as a "return-versus-attrition function" for the duel. Multiple aircraft raid models are developed to determine which point on the return-versus-attrition function is the best operating point for attacking the target. These raid models assume that the aircraft in the raid make stochastically independent, statistically identical attacks. By using the return-versus-attrition function from a single aircraft duel model, and considering the probabilistic survival of area defenses, the optimum raid size and the best policy for the duel are determined. This determination minimises the expected value of the number of aircraft lost in achieving a required level of return to the attackers.

A multiple aircraft raid on multiple targets is considered by starting from the previously stated basic assumption. Here, the problem is to allocate a given number of aircraft among targets and specify the policy for each duel to maximize the total utility to the attackers subject to constraints on the number of aircraft available, the expected

value of the number of aircraft lost, the number of passes an aircraft can make against each target, and the number of weapons an aircraft can carry to each target.

The question of what tactic to choose in the face of uncertainty as to the true parameter values is approached by associating a range of uncertainty with each of the input parameters. We assume complete ignorance of the value that the parameters might take within their respective ranges of uncertainty. A systematic method is developed that aids the decision maker in choosing a nominal set of input values such that the solution that is optimum for those nominal input values constitutes a rational tactic selection considering that the realized or actual input values might fall anywhere within their ranges of uncertainty.

ACKNOWLEDGEMENTS

The author is indebted to many persons both at The Ohio ate University and at Wright-Patterson Air Force Base whose support nd encouragement have made this work possible. In particular, I w h to express my sincere gratitude to Professor William T. Morris for his guidance and encouragement as my adviser, and to Professors Albert B. Bishop and Richard L. Francis who read the manuscript nd made many helpful suggestions. Use of the terms "Markov state" and "composition," as introduced in Chapter II, was suggested by Dr. Francis.

I am grateful to the Air Force for having sponsored my g duate studies at The Ohio State University. It is impossible to ment n all of the persons at Wright-Patterson Air Force Base who have infl nced this work. I must, however, acknowledge the effective and time computer programming support of Mr. Eugene Guthrie of the Direc rate of Computation Services, the fine efforts of Mr. George Cumming n of the Deputy for Development Planning who drew the illustrations, nd the excellent typing of Mrs. Frances Jarnagin.

Finally, I must thank my wife, Harriett, who made this w k possible by accepting a large part of the responsibility of rai ng two small children during the approximately four years while I p epared for and carried out this work.

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PUBLICATIONS

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FIELDS OF STUDY

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Studies in Mathematical Programming. Professors Albert B. Bishop and Richard L. Francis

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GLOSSARY OF SYMBOLS

A - Used to symbolize that target acquisition has occurred.

A^* - Used to symbolize that target acquisition has not occurred.

A_o - The event that target acquisition occurs on an initial pass.

A_o^* - The complement of A_o.

A_oD - The event that acquisition and delivery occur on an initial pass.

A_oD^* - The event that acquisition occurs and weapon delivery does not occur on an initial pass.

A_1 - The event that target acquisition occurs on a subsequent pass.

A_1^* - The complement of A_1.

A_1D - The event that acquisition and delivery occur on a subsequent pass.

A_1D^* - The event that acquisition occurs and weapon delivery does not occur on a subsequent pass.

α - A control variable associated with W.

α_j - A control variable associated with w_j.

α' - The control variable associated with W'.

C - Aspiration level in a P_C duel;
Saturation level in an E_D duel.

C_R - Expected hits required per raid.

$C_R(\alpha;\alpha')$ - Expected hits per raid versus α when using the tactic that is based on the input values associated with α'.

$C_{Rt}(m_t)$ - Expected hits per raid on target t under attack policy m_t.

D_n - Decision vector at stage n.

D_t - The decision vector (R_t, m_t).

d_n - A component of D_n; number of weapons allocated to pass n.

d_n^* - Best salvo size at pass n.

Δs - Increment of variation for s.

$E_R(m)$ - Expected losses per raid when the m^{th} attack policy is used.

$\bar{E}_{Rt}$ - Constraint on the expected losses in attacking targets 1,---,t.

$\bar{\bar{E}}_{RT}$ - An arbitrary limiting value associated with the set S_E.

$F(\alpha;\alpha')$ - A measure of system performance versus α when using the tactic that is based on the input values associated with α'.

$f_n(X_n)$ - Maximum n stage return as a function of x_N.

$f_{ni}(X_n')$ - Maximum expected value of the n stage return as a function of i and X_n'.

$f_t(X_t)$ - Maximum utility achievable in attacks on targets 1,---,t as a function of X_t.

$g_n[X_n, D_n, f_{n-1}(X_{n-1})]$ - n stage return function.

$g_{nij}[X_n', D_n, f_{n-1,j}(X_{n-1}')]$ - n stage return function associated with transition from state i to state j.

$h(\ell;d_n)$ - Probability function of number of hits for a salvo of size d_n.

I - Maximum value of i.

i - Markov state index; i = 1,---,I.

J - The number of components of W.

$K_D(m)$ - Probability of achieving at least one hit per duel when the m^{th} attack policy is used.

$K_{Dt}(m_t)$ - Probability of at least one hit per duel with target t under attack policy m_t.

K_R - Required probability of getting at least one hit per raid.

$K_R(\alpha;\alpha')$ - Probability of at least one hit per raid versus α when using the tactic that is based on the input values associated with α'.

$K_{Rt}(m_t)$ - Probability of at least one hit per raid on target t under attack policy m_t.

k_n - A component of D_n; mode of attack index at pass n.

k_n^* - Best mode of attack at pass n.

L_R - Expected losses per raid when the optimum attack policy (m^*) is used.

$L_R(\alpha;\alpha')$ - Expected losses per raid versus α when using the tactic that is based on the input values associated with α'.

L_T - Expected losses in killing a target with repeated raids when the optimum attack policy is used.

λ_t - The relative importance of target t.

m_t^* - Optimal attack policy for target t.

n - Stage index; $n = 1,---,N$.

$p_{ij}(D_n)$ - Markov state transition probability.

$\pi_n(i;\theta)$ - The probability function of i with parameter θ.

$\phi_{ni}(x_n,s_n)$ - The actual probability of surviving passes n,---,1 versus i, x_n, and s_n.

- a parameter of the example salvo effectiveness function (see equation (III-16)).

R - Number of aircraft per raid.

R^* - Best raid size ($= R(m^*)$).

R(m) - Raid size required to realize C_R or K_R as appropriate when the m^{th} attack policy is used.

R_t - Raid size for target t.

R_t^* - Optimal raid size for target t.

$\bar{R}_t$ - Constraint on the total aircraft allocated to attacking targets 1,---,t.

$r_H(d_n)$ - Expected hits per salvo of size d_n.

$r_K(d_n)$ - Probability of at least one hit in a salvo of size d_n.

$r_n(X_n,D_n)$ - Stage n return versus X_n and D_n.

S_A - Probability of the aircraft surviving the area defenses one way, either from base to target or from target to base.

S_{At} - Probability of the aircraft surviving area defenses enroute to or returning from target t.

$S_D(m)$ - The actual probability of the aircraft surviving the duel when the m^{th} attack policy is used.

$S_{D_n}(X_n)$ - A set depending on X_n where $D_n \in S_{D_n}(X_n)$.

$S_{Dt}(m_t)$ - Actual probability of the aircraft surviving a duel with target t under attack policy m_t.

$S_{\bar{E}_{Rt}}$ - A set where $\bar{E}_{Rt} \in S_{\bar{E}_{Rt}}$.

$S_{m_t}(\bar{E}_{Rt},R_t)$ - A set where $m_t \in S_{m_t}(\bar{E}_{Rt},R_t)$.

$S_{R_t}(\bar{R}_t)$ - A set where $R_t \in S_{R_t}(\bar{R}_t)$.

$S_{\bar{R}_t}$ - A set where $\bar{R}_t \in S_{\bar{R}_t}$.

S_T - Probability of aircraft survival to the point of weapon release on a pass.

S_u - Conditional probability that the aircraft survives a pass given that it survives to the point of weapon release.

S_{X_n} - An arbitrary set where $X_n \in S_{X_n}$.

$S_{s_n}(x_n)$ - A set depending on x_n where $s_n \in S_{s_n}(x_n)$.

$\bar{s}$ - An arbitrary limiting value associated with S_{s_n}.

s_n - Constraining probability of the aircraft surviving passes n,---,1.

t - Index on target when multiple targets are considered. t = 1,---,T.

$t_n(X_n,D_n)$ - State vector transformation.

θ - A parameter of the example salvo effectiveness function (see equation (III-18)).

$U(i)$ - Utility level or damage level associated with Markov state i.

$\bar{U}_T$ - Utility of all raids on the target complex.

$U_t(Z)$ - Utility of Z expected hits on target t.

$U_t^K[p]$ - Utility associated with probability p of killing target t.

$u_D(m)$ - Expected hits per duel when the m^{th} attack policy is used.

$u_{Dt}(m_t)$ - Expected hits per duel with target t under attack policy m_t.

W - Vector of inputs to a raid model.

W' - A vector of input values.

W_o - Vector of optimistic input values.

W_p - Vector of pessimistic input values.

w_j - The j^{th} component of W.

w_{oj} - The j^{th} component of W_o.

w_{pj} - The j^{th} component of W_p.

X_n - State vector at stage n.

X_n' - Residual state vector.

X_t - The state vector $(\bar{R}_t, \bar{E}_{Rt})$.

$\bar{x}_N$ - An arbitrary limiting value associated with S_{x_n}.

x_n - A component of X_n and X_n'; number of weapons available for passes n,---,1.

CHAPTER I

INTRODUCTION

The Problem

General Statement

This work deals with the problem of making a rational selection of tactics for air-to-ground attack when faced by uncertainty as to the exact conditions that prevail. The intention is to provide a systematic method for making the best use of available information. The result is not an "automatic tactics selection" but rather it is a quantitative theoretical structure that can serve as a framework within which to evaluate the multitude of tangible and intangible factors that must be considered in planning an air strike.

Terminology and General Concept

To begin the discussion we will establish some terminology and a general concept. A "raid" will denote a multiple aircraft attack against a target. The raid is composed of "sorties," where each sortie involves one aircraft which takes off, proceeds to the target, takes part in the attack on the target, returns to its base, and lands. When an individual aircraft reaches the target area, its encounter with

the target and target defenses will be referred to as a "duel."[1] A duel may involve multiple passes by the aircraft. A duel might be further described as follows.

A military aircraft with a given number of bombs on board is to attack a defended target. A maximum of N passes can be made subject to fuel limitation or arbitrary policy. A given pass may include acquiring the target, surviving to the point of weapon release, releasing weapons, and surviving the pullout. Some important considerations are as follows.

When the aircraft makes a pass, target acquisition is thought of as an event that has occurred when the pilot has sufficient information to allow weapons to be delivered. This could imply that he visually sees the target or a designated aim point. It could also imply that some sensor such as a radar has produced a desired response. Once target acquisition has occurred, the aircraft is maneuvered into position and aligned for weapon delivery. Because of constraints on the aircraft maneuver capabilities and limitations on pilot reaction time, it is possible for target acquisition to occur too late to allow for a weapon delivery on the same pass.

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[1]The word "duel" when used in this work has a slightly modified meaning from the more conventional use of the word. In a classical duel, as described, for example, by Williams and Ancker (26) the two duelists fire at each other until one is killed. In the duels described herein, an aircraft attacks a ground target while the target defenses fire at the aircraft. All of our duels end if the aircraft is killed. Some of our duels end if the aircraft achieves a specified objective and others of our duels proceed independent of the success of the aircraft's attack on the target.

We will refer to the following two types of passes. An "initial" pass is made when target acquisition has not occurred during any previous pass of the same duel. A "subsequent" pass is made when target acquisition has occurred on at least one previous pass of the same duel. We distinguish between an initial pass and a subsequent pass because target acquisition may be less difficult if it has occurred on a previous pass of the same duel.

The probability that acquisition occurs on an initial pass will be symbolized $P(A_0)$ and the probability that target acquisition occurs on a subsequent pass will be symbolized $P(A_1)$. The symbols $P(A_0D)$ and $P(A_1D)$ will denote the probability that acquisition and delivery occurs on an initial and subsequent pass, respectively.[2]

The notion of aircraft survival and its relation to the capability of the aircraft to attack the target is complicated by the variability in possible damage to the aircraft. "Damage" may result in anything from immediate disintegration of the aircraft to a slight degradation of performance or even to no effect on the aircraft capability. We will make an abstraction of the survival aspect of the problem by defining the probability S_T as the probability of surviving to the point of weapon release. We assume if the aircraft survives, its performance is completely unaffected and if it does not survive, then the aircraft does not participate further in the attack and will be considered as a loss. Also, we define S_u as the conditional probability of the aircraft surviving the pass given that it survives to the point of weapon release.

[2] If B denotes an event, the notation P(B) will denote the probability of occurrence of the event B.

The third principal aspect of a pass concerns weapon effectiveness. The group of bombs delivered on a given pass will be referred to as a "salvo." We can visualize the individual impact points of bombs in a salvo as clustering around the effective aim point for that salvo.[3] The location of the effective aim point varies with respect to the target according to some probability distribution. The function r(d) will represent the effectiveness of a salvo as a function of salvo size, d. It is not the purpose of this work to derive the function r(d), but it is important to note that r(d) is in general a monotonically nondecreasing, concave function of d. In other words, larger salvos have greater effectiveness but there is a diminishing marginal return as salvo size increases.

The term tactics when used in this report includes the following. The operational planner must specify the number of aircraft per raid or "raid size." A policy must be established as to the maximum number of passes per duel and the allocation of weapon load among those passes. A policy must also be established to tell the pilot how to make each pass, i.e., high level, low level, dive, etc. All these items must be specified in a rational way in the face of uncertainty as to the exact conditions that prevail.

---

[3]The variation of individual impact points from the aim point may be unintentional as would be caused by factors such as ballistic dispersion. The variation of individual impact points from the aim point may also be intentional as would be caused by introduction of a systematic delay in the time of release for the various weapons. This latter is generally referred to as "stick bombing," however, we will refer to any group of bombs released on a pass as a salvo.

A rational selection of tactics must involve a criterion or objective. The specific objective can vary from one problem to another. The basic approach adopted in this report is to plan the raid so as to minimize the losses sustained in achieving a given level of effectiveness. A number of measures of effectiveness are considered. For some targets, the level of damage is proportional to the number of hits achieved. This might be true of a large area target. Another common class of targets consists of those such as a revetted artillery site for which a direct hit will deactivate the site and any miss will probably leave the site unharmed. For these targets, the level of effectiveness is in terms of the probability of at least one hit. We can conceive of another class of targets which in itself seems to be of largely theoretical interest but is worth including because it makes a convenient introduction to the most general case. For these targets achieving C hits is adequate; there is no additional value in achieving more than C hits, and achieving less than C hits is of no value. All of the preceding measures of effectiveness are special cases of the general case where the level of damage depends on the number of hits achieved according to some arbitrary utility function. The implications of these various measures of effectiveness are discussed in later chapters.

Motivation

User Oriented

The motivation for this work comes from two sources. The most obvious is the user of tactical air-to-ground weapons systems. As a highly technological nation, we tend to invent new hardware items to

meet military problems. It seems clear that research should also be devoted to learning how to more effectively use existing equipment. This is one of the goals of this research.

Evaluating System Designs

A second and perhaps more notivation for this work comes from the need to perform comparative evaluation of new weapon system designs. This is the aspect from which the weapon system designer views the problem. In evaluating competing new designs or modifications to existing systems, many factors must be considered. These include cost, operational effectiveness, maintenance implications, logistics implications, training implications, and delivery schedule. This report deals with operational effectiveness.

A Two Stage Decision Problem

One way to compare the operational effectiveness of alternative system designs is to formulate a two stage decision problem. At the first stage is the aggregation of decisions that determine the characteristics of the weapon system. These decisions will be referred to as design decisions. They are thought of as being made by a largely fictitious individual to be referred to as the "designer." At the second stage, we consider the operational use of the system that is the product of the first stage. The second stage decisions are made by the "user."

The designer's decision problem at the first stage can usefully be abstracted in the terminology of Luce and Raiffa (19) as an individual decision under uncertainty. The states of nature, θ_j, where

j is an integer such that $1 \leq j \leq J$, are identified as an exhaustive and mutually exclusive set of situations that represent the prospective usage of the system. Various designs a_i where $1 \leq i \leq I$ constitute the alternatives under consideration. A utility u_{ij} is associated with the employment of design alternative i in situation j.

If a subjective probability distribution p_j where $1 \leq j \leq J$ can be defined over the θ_j, then the most desirable design alternative can be chosen by maximizing the expected utility or by some other means based on probabilities, i.e., by making a decision under risk. If the designer is completely ignorant of the probabilities p_j, or chooses to ignore any such information that he may have, then the design decision is made as a decision under uncertainty. Some principle, such as maximin utility, minimax regret, the principle of insufficient reason or the pessimism-optimism index, might be applied. Whether or not the subjective probability distribution can be defined, a key element of the designer's decision is the set of utilities, u_{ij}.

To determine the u_{ij} values for the first stage decision, the second stage decision, i.e., the user's decision, must be considered. The utility of a given system design in a particular situation depends on the design of the system, the objectives toward which use of the system is directed, the nature of the situation, and the manner in which the system is used. The user's decision problem is visualised as a constrained optimization in which tactics are selected to maximize the utility (u_{ij}) within the constraints imposed by the system design (a_i) and the situation (θ_j)'. As an example, the user might wish to maximize the probability of kill. The type of aircraft and

mode of usage dictate the fuel capacity and rate of fuel consumption. The situation controls the base to target distance. These factors combine to provide a constraint on the number of passes that can be made. Thus, the user must maximize the probability of kill subject to a constraint on the number of passes.

The Implications of Tactics Selection

The operational effectiveness of a weapon system in a particular situation can be greatly influenced by the choice of tactics. It is therefore important when evaluating a weapon system to use the tactics that are best for that particular design and situation. For example, consider a night attack. Suppose two design alternatives are being compared and that they are identical with the following exceptions. Suppose design A provides an additional special sensor that has a highly accurate target locating capability but has a short range. Suppose the alternate design B simply provides one additional bomb and relies entirely on the aircraft's radar for target acquisition. Much current practice is to make only one pass at the target per sortie. If the comparison were made on this basis, the special sensor of design A may be useless because its short range means that the information it provides comes too late to be useful. Thus, since design B provides an extra bomb and the designs are the same otherwise, design B will look better. If, on the other hand, each design is used with its own best tactics, the tactic for design A might be to use the special sensor to locate the target on the first pass and then deliver weapons on the second pass. Design B might still call for just one pass. On this

basis, design A may or may not look better than design B but the comparison certainly seems more reasonable.

The principle that each design alternative should be evaluated using its own best tactic is not new. The difficulty is determining what are the best tactics. Determining the best tactic for a given system design in a particular situation can be a major constrained optimisation problem. The objective function and the constraints must be carefully formulated and a solution must be found.

When evaluating a system design, it is sometimes tempting to bypass the tactics optimisation problem by having a "panel of experts" choose the tactics appropriate for each situation in which the system is to be evaluated. It might then be argued that if all designs are evaluated using the same tactic in a given situation, the comparison is "fair." This is simply not true as was qualitatively indicated in the example. It might also be argued that the panel of experts can assign each system its own best tactics. This approach has a number of limitations. The designs being evaluated are generally different from existing equipment and the situations of interest are usually beyond the experience of any panel of experts. Further, even when applicable experience is available, existing practice is not necessarily optimal. Finally, the limitations of people in judging the implications of complex quantitative relationships are well known.

The tactics optimization problem seems to be inseparable from the design evaluation problem whenever the user has some latitude of choice as to the manner of system employment. In a sense, we might view the role of the designer as that of establishing constraints

within which the user must operate. Thus, selecting a best system design is in effect a problem of deciding which set of alternative modes of use should be provided to the user. The cost of the system generally increases as the variety of modes of use available to the user is increased.

The Implications of User Uncertainty

Whether we are considering the user's problem or the design evaluation problem which contains the user's problem, a considerable complication is introduced by the user's uncertainty. The prospective usage of the system is characterised by an array of situations. Determining the utility of a given system design in a particular situation can be treated as a conceptually simple constrained optimisation problem if the situation is exactly defined. Unfortunately, the situation is not exactly defined in actual practice. Such quantities as the probability of aircraft survival during a phase of the sortie are generally matters of considerable uncertainty. Thus, the specification of a situation must generally reflect the degree of uncertainty involved when values are given for the characterising parameters.

From the standpoint of the user, a systematic method should be available for considering the implications of uncertainty and his options to control the outcome by appropriate tactics selection. From the standpoint of the designer, it is important to understand the implications of user uncertainty because it affects the utility of a system design in a particular situation. A general principle might be that there is no point in providing the user with options that he can't use effectively because he is uncertain of the situation.

Research Philosophy

A Concept of Operations Research

One view is that operations research is basically a conjec ural discipline which involves investigating the consequences of assum tions. Accordingly, three aspects of this discipline might be identified First is the specification of assumptions, second is the developm it and employment of investigative techniques, and third is the appl ca- tion of theoretical results in making judgments about practical r l world problems. All or some combination of these aspects may be - propriate to a given study.

Assumptions are invariably involved in any operations rese investigation. These assumptions should be such that knowledge of their consequences has some value. If they lead to a theoretical structure that parallels some real world situation, then perhaps i - ferences can be drawn about relationships and the consequences of ts in the real world by studying the corresponding relationships in t theoretical structure. The notion is that studying the theoretica structure is more convenient and cheaper than studying the corresp d- ing real world situa ion. This is particularly true of military problems.

The investigative techniques used in a study can range from operational experiments to mathematical analysis. We might think c an operational experiment as an attempt to establish a relationship between the theoretical structure and the real world, i.e., hypothe es or results deduced therefrom are tested. At the other extreme, mat - matical analysis deals entirely with logical relationships. Betwee

these methodological extremes are investigative techniques such as gaming and simulation.

Using theoretical results to make judgments about real world problems is the payoff and presumably the purpose of operations research activity. Whether this judgment is made by the operations researcher or by someone else is not a central issue. The important point is that some form of application should be made. In this regard the following quotation from Flagel, Huggins, and Roy (6) is pertinent: "The characteristic of a true operations research study is that it provide realistic answers to an actual practical problem. In this context the tools and techniques used should never be limiting; the goal is to select techniques that allow all significant factors of the actual problem to be considered."

The Nature of this Work

In this work, assumptions are set forth that lead to a structure resembling a particular real world military situation. Recursive analysis methods and some notions from decision theory are used to determine and express the consequences of these assumptions. The purpose is to explicitly define the methods that are used, show how they apply in deducing the consequences of a particular set of assumptions, and quantitatively illustrate the nature of the consequences by way of numerical examples.

This is a theoretical study that points out how a given type of structure can be usefully analysed. It therefore seems important that we seek generality and flexibility. The results should show how to

approach a variety of related problems. We seek problem solving techniques more than specific analytical or numerical results.

It is not our purpose to make judgments about specific real world problems. This is properly the province of those who are directly involved in the activities that are being studied. This might be the concern of an operations research activity that is an integral part of the military organization that is conducting the activities of interest. A certain amount of foundation can be laid in an isolated academic environment but if the model is to finally be truly effective, it and the research that goes along with it must become a part of the using organization. In a sense, the one who makes the application must also be a researcher. He must modify and continually develop the theory.

Consider the notion of model "validation" in the light of the preceding discussion. The model would be accepted as valid if its structure parallels the real world situation of interest sufficiently accurately to allow useful conclusions to be drawn. The decision maker must decide whether or not this is true of a given model in a given decision situation. Accordingly, the validity of an operations research model is meaningful only when a specific decision is to be made; otherwise, there is no basis for judging whether or not the model constitutes a satisfactory representation and the concept of validity has no meaning. Since we are not making decisions, model "validity" will not be of concern. We should, however, be concerned with the logical consistency of the theoretical structure. We may also be concerned that real world situations exist for which some decision maker might be willing to consider our structure as a "valid" representation when

making practical decisions. This last thought emphasizes the importance of generality and flexibility in our methods.

The approach that is developed in this report is basically numerical. Some analytical relationships are investigated but the complexity of the problem along with the critical need for flexibility of the model seem to dictate a largely numerical approach. It should be noted, however, that the recursive analysis techniques that are used are subtile and powerful computational tools. The numerical character of these techniques seems to pose no great practical problem since computers are widely available to all potential users.

Related Literature

The air-to-ground attack problem as treated in this work does not seem to be a very popular subject in the open literature. There are, however, articles which relate to various aspects of our problem. The most applicable of these will be cited and categorized with some indication of how they relate to our subject.

Stochastic Duels

Williams and Ancker (26) have developed a theory of "stochastic duels" which they describe as follows:

> In the "fundamental" duel, two duelists, A and B, fire at each other until one is killed. A's firing time (that is, the time between rounds) is a random variable with a known probability density function, $f_A(t)$. Successive firing times are selected from $f_A(t)$, independently and at random. The situation is the same for B except that his firing time has a different density function, $f_B(t)$. Each time A fires, he has a fixed probability, P_A, of killing B. Similarly, B's kill probability is P_B. After the starting signal, each contestant loads, aims, and fires the first round. That is, in the "fundamental" duel, they start with unloaded weapons. Both A and B have unlimited supplies of

ammunition that, among other things, makes a kill a certainty. A wins if he is the one to score a kill. The probability of this will be called P(A) and $P(A) + P(B) = 1$.

In reference 26, P(A) is determined for the case where $f_A(t)$ and $f_B(t)$ are negative exponential distributions. The effect of giving one contestant a random initial time advantage is also investigated. In reference 3, Ancker extends the model to the case where both contestants have limited ammunition supplies. He determines P(A), P(B), and the probability that they both run out of ammunition. Both of these papers approach the problem by computing the distribution of time to kill for the two contestants and then determining the probability that one gets a kill before the other does.

In reference 4, Ancker and Williams consider the fundamental duel with discrete firing times where A and B fire at fixed but possibly different intervals and the ammunition supply is unlimited. They also consider a case where the contestants fire simultaneously, a near miss by one causes the other to lose one firing turn, and ammunition is unlimited. Finally, this paper considers duels that are not one-on-one.

Our air-to-ground attack problem might be considered as a type of duel where the contestants fire at each other simultaneously each time the aircraft makes a pass. If the aircraft is contestant A and the target with its defenses is contestant B, we are interested in P(A), the probability that the aircraft kills the target before the aircraft is killed. In our duel, however, contestant A has a limited ammunition supply which he can fire in salvos with arbitrary salvo effectiveness

versus salvo size. We are interested in maximizing P(A) and determining the best allocation of weapons among passes on the part of contestant A. This type of duel is developed in Chapter V. Our generalization of the above type of duel, which is also developed in Chapter V, extends the aircraft's options to include multiple modes of weapon delivery and it considers the aircraft's target acquisition problem. A further complication of our duel is that simply maximizing the aircraft's probability of kill without regard for its probability of survival is not appropriate. Because of this, we maximize the aircraft's probability of kill subject to a constraint on its probability of survival.

Some further papers on stochastic duels that appear in the literature involve the distribution of the number of rounds fired, reference 1, and the distribution of the time duration, reference 2.

Tactical Air Games

Fulkerson and Johnson (16) formulate the following tactical air game in which each side must continually allocate available aircraft between counter air and ground support missions. They describe the formulation as

> . . . a multi-move game in which both sides, at each period of the campaign, simultaneously deploy their forces between the two missions. Each force suffers a fixed rate of attrition per period due to accidents, etc., and in addition loses planes proportionally to the size of the enemy's attack on his air fields. Replacements for each side are received periodwise, and these may be functions of time. The payoff is assumed to be the difference between the total number of ground support sorties flown by the two sides during the campaign, discounted for future time periods.

The symmetric case in which the attrition rates are the same for both sides is solved for both finite and infinite campaigns.

Bellman and Dreyfus (7) show how to treat the same problem by using dynamic programming. The model is further discussed and developed by Weiss (25) and by Berkovitz and Dresher (8,9).

Two essential inputs for this model are the red and blue kill potential per plane sent against the opponent's air fields. For the symmetric case, these are equal. Our model, particularly the duel of Chapters III and IV (called herein the E_H duel) should be helpful in evaluating these quantities. Furthermore, once the number of ground support sorties to be flown has been determined at a given stage of the game, it is necessary to allocate those sorties among the prospective targets and determine the best tactics for each raid. This problem is treated in Chapter VII of this report.

Allocation of Weapons[4]

Manne (20) discusses the problem of allocating a number of weapons to a complex of targets. He reduces this problem to the form

$$\text{Minimize} \quad \sum_{j=1}^{n} a_j \, (1 - p_j)^{y_j} \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_j = m \tag{2}$$

$$y_j \geq 0 \qquad j = 1, \cdots, n \tag{3}$$

where $j = 1, \cdots, n$ = the index on targets

a_j = the unit worth of target j

[4]The notation used in this section is in keeping with the weapon allocation literature and is generally different from that in the rest of the paper which is based on dynamic programming literature, i.e., Ref. 17 and Ref. 22.

p_j = the probability of kill for any weapon versus target j

y_j = the number of weapons assigned to target j.

He then shows how to formulate the above as a transportation problem which results in an integer solution.

G. G. den Broeder, Ellison, and Emerling (10) also consider the weapon allocation problem and prove some applicable theorems. Using the above notation where it is applicable, they first assume that $p_1 = p_2 = \cdots p_n = p$ and consider the form

$$\max V = \sum_{k=1}^{n} V_k \alpha_k (y_1, \cdots, y_n) \tag{4}$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_j = m \tag{5}$$

$$y_j \geq 0 \qquad j = 1, \cdots, n \tag{6}$$

where V = expected value of the targets destroyed

V_k = the value of destroying exactly k targets

$\alpha_k(y_1, \cdots, y_n)$ = the probability of destroying exactly k targets as a function of the allocation $(y_1, \cdots, y_n)$.

For this problem, they prove the following theorems I and II.

> "Theorem I. If the V_k are nondecreasing functions of k, then the maximum V is attained when the y_j's differ by at most one."

> "Theorem II. The probability, P_k, of destroying k or more targets is, for each k, a maximum when the y_j's differ by at most one."

They also consider the problem defined by equations (1), (2) and (3). For that problem, they prove that the following theorem III holds for all p_j.

"Theorem III. Given that $\left\{ \overline{y}_j \right\}$ minimizes

$$a(m) = \sum_{j=1}^{n} a_j \, (1 - p_j)^{y_j} \tag{7}$$

$$\text{subject to } \overline{y}_j \geq 0 \tag{8}$$

$$\text{and } \sum_{j=1}^{n} \overline{y}_j = m \tag{9}$$

then $\left\{ \hat{y}_j \right\}$ minimizes

$$a(m+1) = \sum_{j=1}^{n} a_j \, (1 - p_j)^{y_j} \tag{10}$$

$$\text{subject to } \hat{y}_j \geq 0 \tag{11}$$

$$\text{and } \sum_{j=1}^{n} \hat{y}_j = m + 1 \tag{12}$$

if $\hat{y}_j = \overline{y}_j$ for $j \neq k$ and $\hat{y}_k = \overline{y}_k + 1$, where k satisfies

$$a_k \, (1-p_k)^{\overline{y}_k} \, p_k = \max_{1 \leq j \leq n} \left\{ a_j \, (1-p_j)^{\overline{y}_j} \, p_j \right\} \text{"} \tag{13}$$

According to the authors, if one interprets $a_k \, (1 - p_k)^{\overline{y}_k}$ as a revised estimate of the value of the k^{th} target based upon an optimum assignment of m weapons, the procedure implicit in theorem III merely states that an added weapon should be assigned to that target for which the

expectation of (revised) value destroyed is largest. Thus, by starting with $m = 0$, weapons can be added one at a time in such a way that the allocation at each stage is optimal.

Lemns and David (18) extend the solution to the case where there is more than one type of weapon available. Again, using the foregoing notation where it is applicable, their problem takes the form

$$\text{Maximize} \quad \sum_{j=1}^{n} a_j \left[1 - \prod_{i=1}^{m} (1 - p_{ij})^{y_{ij}}\right] \tag{14}$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_{ij} = m_i \qquad i = 1,\text{---},m \tag{15}$$

$$y_{ij} \geq 0 \qquad i = 1,\text{---},m \quad j = 1,\text{---},n \tag{16}$$

where p_{ij} = probability of kill for weapon type i versus target j

y_{ij} = number of weapons of type i allocated to target j

m_i = number of type i weapons that are available.

Their approach is to determine values L_i which represent the number of type 1 weapons required to be the equivalent of one type i weapon. By using this device, the problem reduces to the form of equations (1), (2) and (3) that was considered by the previous two references. Lemns and David indicate the possibility of solving the problem by the two previous methods which both produce integer results. They also offer a solution that they indicate was obtained by the method of Lagrange

multipliers. This method treats the weapons being allocated, i.e., the equivalent number of type 1 weapons, as a continuously variable quantity. The answer may involve fractional numbers of weapons. Because of round-off error, this approach would be most applicabl if the number of weapons being allocated was large compared to the number of targets.

The weapon allocation papers that have been discussed can be related to the air to ground attack problem as follows. Suppose we wish to maximise the expected value of the number of hits. We might do this if the attacker's utility is a linear function of the number of hits. Thus, we have the problem:

$$\text{Maximise} \quad E = \sum_{j=1}^{n} S^j \, r_j(y_j) \tag{17}$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_j = m \tag{18}$$

$$y_j \geq 0 \qquad j = 1,\text{---},n \tag{19}$$

where $j = 1,\text{---},n$ = index on air to ground pass

S = probability of the aircraft surviving a pass

y_j = the number of weapons delivered on the j^{th} pass

$r_j(y_j)$ = the expected value of the number of hits on pass j as a function of the number of weapons delivered on pass j.

m = the total number of weapons carried by the aircraft.

Assuming stochastically independent delivery errors for the weapons that are delivered on pass j, we have the special case where

$$r_j(y_j) = 1 - (1 - p_j)^{y_j} \tag{20}$$

and equation (17) becomes

$$\text{Maximize} \sum_{j=1}^{n} S^j \left[1 - (1 - p_j)^{y_j}\right] \tag{21}$$

which has the same solution as

$$\text{Minimize} \sum_{j=1}^{n} S^j (1 - p_j)^{y_j} \tag{22}$$

which the same as (1) if a_j is interpreted as S^j.

The problem stated in equations (17), (18) and (19) is essentially the same as the simple E_H duel of Chapter III. The recursive analysis technique that is used in this work offers a practical way of obtaining the solution with no special restrictions on the form of the function $r_j(y_j)$ although a form similar to that given by equation (20) is used for the numerical examples. In addition, the use of recursive analysis makes it practical to consider probabilistic target acquisition, multiple modes of attack, and a constraint on the probability of the aircraft surviving the duel. Finally, the recursive techniques allow solution of the problems that are discussed in Chapters V and VI which are not treated in the foregoing articles.

Sequential Decision Processes

The basic purpose of this investigation is to show how to usefully analyze problems related to the selection of tactics for air-to-ground attack. Dynamic programming is an extremely useful technique for obtaining solutions to these problems. The dynamic programming theory that is discussed and used herein was adapted or developed only as needed to solve the specific problems at hand. These problems involve sequential decision processes with a relatively small number of stages.

The "principle of optimality" as introduced by Bellman (5) is the starting point for developing all of the recursive relationships that are discussed herein. The technique for applying the principle of optimality to solve deterministic sequential decision problems is discussed by many authors including Bellman (5) and Nemhauser (22), Chapter II. Howard (17) extends the application of this principle to the solution of sequential decision problems involving Markov processes. He provides for selecting, at each stage, the best act from an array of alternative actions. Nemhauser (22), Chapter V, discusses a still more general form of multistage decision model which applies to what he calls a "stochastic optimization problem" or a "multistage optimization under risk." He introduces a random variable, k_n, at stage n whose value determines the stage return and the state variable transformation.[5] He formulates the case where the k_n are stochastically

[5] To illustrate the meaning of these terms, the "stages" generally correspond to passes in our problems; the "stage return" is the utility derived from the weapons delivered on the corresponding pass; the "state variable" is a vector that characterizes the state affairs when the aircraft is preparing to make a pass.

independent from stage to stage. He also points out that if the distribution function of k_n depends on k_{n+1}, the value at the previous stage, the process is Markovian. The probability distribution of k_n may also depend on the values taken by k at all previous stages in which case the process is not Markovian.

The "monotonicity assumption" is basic to establishing the optimality of the dynamic programming results. This assumption was introduced by Mitten (21) and is further discussed by Denardo and Mitten (12) and by Nemhauser (22). The exact statement of the monotonicity assumption varies somewhat with each author making the statement that best serves his own purposes.

Charnes and Schroeder (11) discuss sequential decision processes from the standpoint of multistage games or stochastic games. Their paper relies heavily on the work of Shapely (23). A stochastic game consists of a series of stages where the states occupied by two competing players are subject to probabilistic transition from stage to stage according to transition probabilities controlled jointly by the two players. Associated with each transition is a payoff from player two to player one, i.e., the game is zero sum. A terminating stochastic game is one in which at each stage there is a non-zero probability of the play ending. Charnes and Schroeder, following Shapely's development, show how such a game can be solved by iterative application of linear programming. They also give a stopping criterion for the iterative process.

Of particular interest here is the demonstration by Charnes and Schroeder that when player one knows player two's strategy at each

stage, the stochastic game reduces to the discounted Markovian decision process that is discussed by Howard (17). If the process is truncated so there is a finite number of stages, Howard's value iteration applies. If the process is not truncated so there is an infinite number of stages, Howard's "policy iteration" algorithm applies. Furthermore, Charnes and Schroeder show how linear programming can be used to determine the optimum stationary policy for the infinite stage Markovian decision process.

The technique for applying the principle of optimality to our problems is discussed in Chapter II. The basic philosophy for dealing with multistage decision problems is similar to that of Nemhauser, however, the subject is developed in such a way as to considerably simplify the development of appropriate recursive relationships and hopefully make them highly intuitive. This simplification is important when applying dynamic programming to complex problems. The monotonicity assumption is stated here in such a way as to minimize the difficulties encountered in deciding whether or not it applies in a given situation.

The treatment of Markovian decision processes is developed in Chapter II to the extent of separating the state vector into two parts, one of which is subject to probabilistic transformation while the other is subject to a deterministic transformation that may depend on the outcome of the probabilistic transformation. This innovation results in improved computational efficiency and a reduction in computer memory requirements. It also considerably simplifies the mechanics of applying the resulting recursive relationship to the problem. This technique does not appear in the literature that has been reviewed.

There also appears in Chapter II of this work an "equivalence" assumption. This assumption accompanies the monotonicity assumption in establishing the optimality of the dynamic programming results obtained for Markovian decision problems. This assumption is not made explicit in the literature that has been reviewed, although it is invariably satisfied by the recursive relationships that appear. A probable reason why the equivalence assumption does not appear in the literature is that most discussion of Markovian decision processes is in terms of additive present and future returns (see references 22, 17 and 11). In that case, the equivalence assumption is satisfied. In this work, we treat the general case where present and future returns are not necessarily additive (see Chapter V) so the equivalence assumption becomes important.

CHAPTER II

GENERAL RECURSIVE RELATIONSHIPS

Analysis of Multistage Systems

The purpose of this chapter is to develop and discuss some generalized recursive relationships that will be used in later chapters. This discussion will introduce some terminology and notation. It will also make explicit the requirements that must be met if these recursive relationships are to serve our purposes.

We are interested in making decisions relating to the performance of serial multistage systems. Nemhauser (22) defines such a system as "a set of stages joined together in series so that the output of one stage becomes the input to the next." Our problems involve a finite, relatively small number of stages, i.e., an infinite stage approximation as introduced by Bellman (5, p. 11) is not generally applicable.

Consider a system of stages indexed $n = 1, 2, \cdots, N$ where each stage in the system is characterized as follows. The state of the system at stage n is completely described by the state vector X_n. The decision made at that stage is designated by the decision vector D_n. The return realized depends on the state and the decision and is denoted by the function $r_n(X_n, D_n)$. The output of stage n, which becomes the input state for stage n-1, depends on the state and decision at stage n and is indicated by the following "transformation" relationship.

$$X_{n-1} = t_n\ (X_n,\ D_n) \tag{1}$$

We will assume that t_n (··) is single valued. A system composed of a series of N stages might be illustrated by Figure 1 which is similar to the corresponding diagrams used by Nemhauser (22).

The basic analysis approach to be followed involves composing or "putting together" the system by starting at stage one and adding one stage at a time to the already existing structure. The stages will be numbered according to the order in which they are added in the composition process. The central idea of the composition process can be described as follows.

Suppose we have an existing structure of n-1 stages. Since the state of the system at stage n-1 is completely described by the state vector X_{n-1}, then for a given system structure, no other input information is required to determine the maximum return that is realizable from the existing n-1 stages. Let us designate this maximum return from the n-1 stages as $f_{n-1}\ (X_{n-1})$.

We now wish to expand the existing n-1 stage structure to include stage n. The situation might be illustrated by the diagram in Figure 2. The function

$$g_n'\left[r\ (X_n,D_n),\ f_{n-1}(X_{n-1})\right] = g_n\left[X_n,D_n,\ f_{n-1}(X_{n-1})\right]$$
$$= g_n\left[X_n,D_n,\ f_{n-1}\ (t_n(X_n,D_n))\right] \tag{2}$$

characterizes the composition of the stage n return with the maximum return available from the remaining n-1 stages. This will be referred to as the "return function."

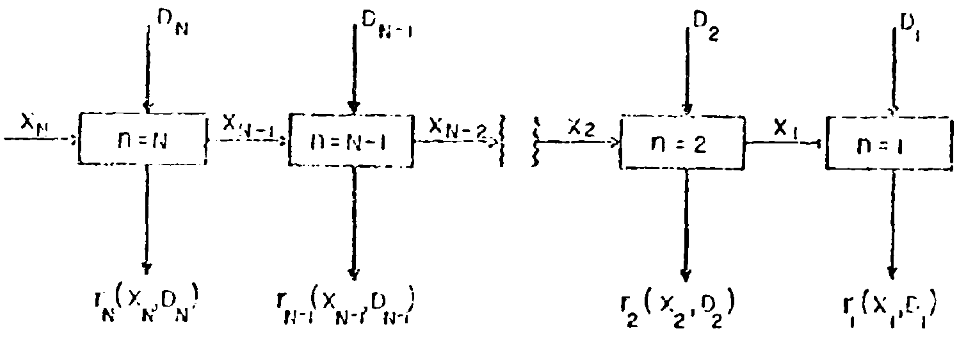

Fig. 1.--Diagram of Multistage Process.

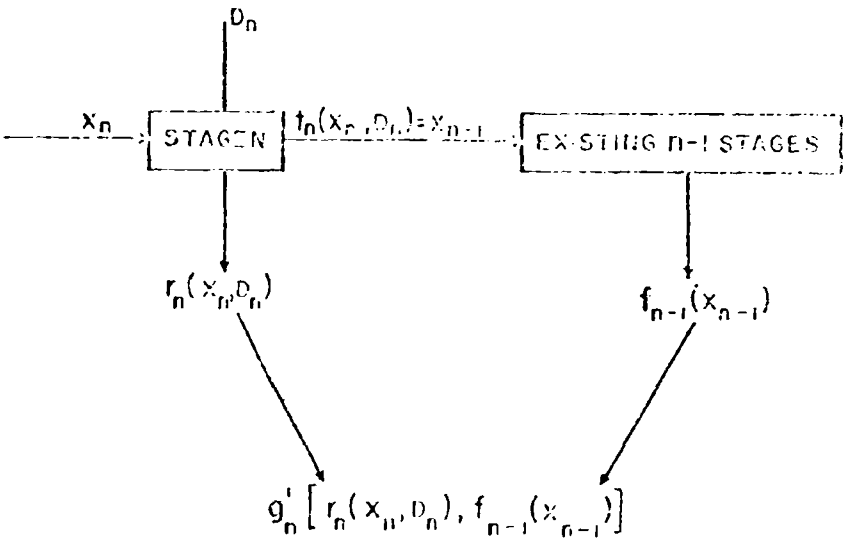

Fig. 2.--The Composition Process.

The maximum composite n stage return is now determined, subject to appropriate conditions, by choosing the value of D_n that maximizes the n stage return function.

$$f_n(X_n) = \underset{D_n}{\text{Max}} \quad g_n\left[X_n, D_n, f_{n-1}(X_{n-1})\right] \tag{3}$$

Bellman and Dreyfus (6) indicate that the basic functional equation of dynamic programming has the form of equation (3).

A Deterministic Decision Process

A deterministic decision process will be defined as one in which the transformation relationship of equation (1) is such that X_{n-1} is deterministically known if X_n and D_n are known. We are interested in specifying and justifying sufficient conditions such that if they are met, recursive application of equation (3) to a deterministic decision process will yield the maximum n stage return and the optimal decision rule at each stage.

Suppose transformation relationships of the form of equation (1) are given. Let S_{X_n} and $S_{D_n}(X_n)$ be sets of all allowable values of X_n and D_n, respectively, at stage n where $S_{D_n}(X_n)$ depends on X_n. Note that these sets must have the property that if $X_n \in S_{X_n}$ and $D_n \in S_{D_n}(X_n)$, then $t_n(X_n, D_n) \in S_{X_{n-1}}$ for all $2 \leq n \leq N$.

Define the one stage return as $g_1(X_1, D_1)$, a function that is defined for all $X_1 \in S_{X_1}$ and $D_1 \in S_{D_1}(X_1)$. Then the function

$$f_1(X_1) = \underset{D_1 \in S_{D_1}(X_1)}{\text{Max}} \quad g_1(X_1, D_1) \tag{4}$$

is defined for all $X_1 \in S_{X_1}$; it is the maximum one stage return.

Now let $f_{n-1}(X_{n-1})$ be the maximum n-1 stage return as a function of X_{n-1} and let it be a function that is defined for all $X_{n-1} \in S_{X_{n-1}}$, $2 \leq n \leq N$. Note that $f_{n-1}(X_{n-1})$ can be visualized as the result of having maximized some objective function relating to the n-1 stage structure subject to the condition that the state of the system at stage n-1 is completely described by the state vector X_{n-1}.

The return function $g_n\left[X_n, D_n, f_{n-1}(X_{n-1})\right]$ characterizes the composition of the return from stage n with the maximum return from the remaining n-1 stages. Suppose the function $g_n\left[\cdots\right]$ has the following property:

> Monotonicity: If $f_{n-1}(X_{n-1})$ is replaced by an independent variable, say y, $g_n\left[X_n, D_n, y\right]$ is defined and it is a monotonically nondecreasing function of y for all $X_n \in S_{X_n}$, $D_n \in S_{D_n}(X_n)$, and $2 \leq n \leq N$.

Then it follows from the definitions of monotonicity and $f_{n-1}(X_{n-1})$ that the function $g_n\left[X_n, D_n, f_{n-1}(t_n(X_n, D_n))\right]$ represents the maximum return that is obtainable from the n stage system for given X_n and D_n.[1] Accordingly, the basic functional equation

$$f\,(X\,) = \max_{D_n \in S_{D_n}(X_n)} g_n\left[X_n, D_n, f_{n-1}(t_n(X_n, D_n))\right] \tag{5}$$

represents the maximum n stage return subject only to the value of X_n; the function $f_n(X_n)$ is defined for all $X_n \in S_{X_n}$, $2 \leq n \leq N$.

[1] A proof of this statement appears in Appendix B.

Thus $f_1(X_1)$ is the maximum one stage return and we have shown that under suitable conditions, if $f_{n-1}(X_{n-1})$ is the maximum n-1 stage return, then $f_n(X_n)$ is the maximum n stage return. So by mathematical induction, $f_n(X_n)$ is the maximum n stage return for all $X_n \in S_{X_n}$ and $1 \leq n \leq N$.

We will now summarise the sufficient conditions for the optimality of $f_n(X_n)$ in a deterministic decision process. For completeness, let

$$g_1 \left[X_1, D_1, f_o(X_o)\right] \equiv g_1 \; (X_1, D_1) \tag{6}$$

for all $X_1 \in S_{X_1}$ and $D_1 \in S_{D_1}(X_1)$.

If the following transformation relationships are given,

$$X_{n-1} = t_n \; (X_n, D_n) \qquad ; \qquad 2 \leq n \leq N$$

there must exist sets S_{X_n} and $S_{D_n}(X_n)$ such that if $X_n \in S_{X_n}$ and $D_n \in S_{D_n}(X_n)$ then $t_n(X_n, D_n) \in S_{X_{n-1}}$ for all $2 \leq n \leq N$. Also, the function $g_1 \left[X_1, D_1, f_o(X_o)\right]$ must be defined for all $X_1 \in S_{X_1}$ and $D_1 \in S_{D_1}(X_1)$. Furthermore, the function $g_n \left[X_n, D_n, f_n(X_{n-1})\right]$ must possess the monotonicity property that was defined previously in this section for all $X_n \in S_{X_n}$, $D_n \in S_{D_n}(X_n)$, and $2 \leq n \leq N$. If the foregoing conditions are met, then recursive application of equation (5) will yield the maximum n stage return for all $1 \leq n \leq N$.

A Generalized Markovian Decision Process

In the deterministic decision process, the transformation relationship, equation (2), relates the state of the system at stage n-1

deterministically to the state of the system, X_n, and the decision, D_n, at stage n. We will now consider the situation where the state of the system at stage n-1 is a random variable with a probability function that depends on X_n and D_n. This conditional probability function will be denoted $p(X_{n-1} \mid X_n;D_n)$. This is a Markov process since

$$p(X_{n-1} \mid X_n;D_n) = p(X_{n-1} \mid X_N,X_{N-1},\cdots,X_n;D_N,D_{N-1},\cdots,D_n)$$

$$1 \leq n \leq N \qquad (7)$$

With this process in mind, suppose that we have an existing n-1 stage structure. Since the process of interest is now a stochastic process, the n-1 stage return for a given value of X_{n-1} is a random variable. Let $f_{n-1}(X_{n-1})$ denote the maximum expected value of the n-1 stage return. We wish to expand the structure to include stage n. In general, the composite return of stage n and the remaining n-1 stages depends on X_n, D_n, and X_{n-1}. Denote this composite return by the return function $g_n'\left[X_n,D_n,X_{n-1},f_{n-1}(X_{n-1})\right]$. Then the maximum expected value of the n stage return, subject to suitable conditions, is given by

$$f\,(X\,) = \max_{D_n \in S_{D_n}(X_n)} \sum_{X_{n-1} \in S_{X_{n-1}}} p(X_{n-1} \mid X_n;D_n)\; g_n'\left[X_n,D_n,X_{n-1},f_{n-1}(X_{n-1})\right] \qquad (8)$$

For our purposes, equation (8) is the basic functional equation that applies to Markovian decision processes.

A Special Markovian Decision Process

We will now discuss a special case of the Markovian decision process that is of particular interest in the applications that are to be considered.

Suppose the components of the state vector can be divided into two sets as follows. Suppose the first set consists of one component, the value of which can be designated by the integer variable i, $1 \leq i \leq I$. The variable i will be said to designate the Markov state of the system (for reasons that will become clear). The remaining components of the state vector completely describe the state of the system in all respects other than the Markov state. The vector composed of these remaining components will be referred to as the "residual state vector" and will be designated X_n'.

Now suppose that transitions among the Markov states are probabilistic and are governed by a stochastic matrix of transition probabilities having elements designated $p_{ij}(D_n)$, i.e., $p_{ij}(D_n)$ is the conditional probability of being in Markov state j at stage n-1 given that the system is in Markov state i at stage n when decision D_n is made. Let the transformation of the residual state of the system be deterministic for given n, i, j. The residual state of the system at stage n-1 is determined by the transformation relation

$$X_{n-1}' = t_{nij}\,(X_n', D_n) \tag{9}$$

Thus, we have probabilistic transition among the Markov states which controls the transformation of the residual state vector.

The foregoing can be related to the basic functional equation (8) as follows. Recalling the notation that was described in the first paragraph of the previous section, the conditional probability function that appears in equation (8) can be expressed as[2]

$$p(X_{n-1} \mid X_n;D_n) = p(X'_{n-1},j \mid X'_n,i;D_n)$$

$$= p_1(X'_{n-1} \mid j,X'_n,i;D_n)\ p_2(j \mid X'_n,i;D_n) \qquad (10)$$

Now considering equation (9),

$$p_1(X'_{n-1} \mid j,X'_n,i;D_n) = 1.0 \qquad (11)$$

if X'_{n-1} is the value given by equation (9) and $p_1(X'_{n-1} \mid j,X'_n,i;D_n)$ is zero otherwise. Thus, equation (10) can be written as

$$p(X_{n-1} \mid X_n;D_n) = p_2(j \mid X'_n,i;D_n) \qquad (12)$$

Now assuming that the value taken by j is stochastically independent of X'_n, i.e., the Markov state at stage n-1 is independent of the residual state at stage n, equation (12) becomes

$$p(X_{n-1} \mid X_n;D_n) = p_2(j \mid i;D_n) \equiv p_{ij}(D_n) \qquad (13)$$

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[2]The quantity $p(X'_{n-1},j \mid X'_n,i;D_n)$ is a conditional joint probability function in the random variables X'_{n-1} and j. The second equality in equation (10) follows from a relation in basic probability theory. If u and v are random variables with the joint probability function h(u,v), then $h(u,v) = h_1(u \mid v)\ h_2(v)$ where $h_1(u \mid v)$ is the conditional probability function of u for given v and $h_2(v)$ is the marginal probability function of v.

Here we have introduced a convention that Markov state variables and stage indices will appear as subscripts while residual state variables and decision variables will appear as arguments. This convention will be followed henceforth.

Since the Markov state variables are to appear as subscripts, the factor $g_n'[\cdots\cdot]$ that appears in equation (8) becomes

$$g_n'[\cdots\cdot] = g_{nij}'[X_n', D_n, X_{n-1}', f_{n-1,j}(X_{n-1}')] \tag{14}$$

and if X_{n-1}' is replaced by its equivalent from equation (9),

$$g_n'[\cdots\cdot] = g_{nij}[X_n', D_n, f_{n-1,j}(t_{nij}(X_n', D_n))] \tag{15}$$

Finally, for this special case, equation (8) becomes

$$f_{ni}(X') = \underset{D_n}{\text{Max}} \left\{ \sum_{j=1}^{I} p_{ij}(D_n)\ g_{nij}[X_n', D_n, f_{n-1,j}(t_{nij}(X_n', D_n))] \right\} \tag{16}$$

Note that the prime on the $g_{nij}'[\cdots\cdot]$ in equation (14) does not appear in equations (15) and (16) because the $g_{nij}[\cdots\cdot]$ that appears in equations (15) and (16) represents a functional form that is different from the functional form represented by the $g_{nij}'[\cdots\cdot]$ that appears in equation (14).

We will now proceed to specify and justify sufficient conditions such that if they are met, recursive application of equation (16) will result in values of $f_{ni}(X_n')$ that represent the maximum expected value of the n stage return.

Suppose transformation relationships of the form of equation (9) are given. Let $S_{X_n'}$ and $S_{D_n}(X_n')$ be sets of all allowable values of X_n' and D_n respectively at stage n, where $S_{D_n}(X_n')$ depends on X_n. These

sets must have the property that if $X_n' \in S_{X_n'}$ and $D_n \in S_{D_n}(X_n')$, then $t_{nij}(X_n', D_n) \in S_{X_{n-1}'}$ for all $1 \leq i \leq I$, $1 \leq j \leq I$, $2 \leq n \leq N$.

Define the one stage return as $g_{1ij}(X_1', D_1)$, a function that is defined for all $X_1' \in S_{X_1'}, D_1 \in S_{D_1}(X_1')$, $1 \leq i \leq I$, $1 \leq j \leq I$. Then the function

$$f_{1i}(X_1') = \underset{D_1 \in S_{D_1}(X_1')}{\text{Max}} \sum_{j=1}^{I} p_{ij}(D_1)\, g_{1ij}(X_1', D_1) \tag{17}$$

is the maximum expected value of the one stage return; it is defined for all $X_1' \in S_{X_1'}$, $1 \leq i \leq I$.

Now let $f_{n-1,j}(X_{n-1}')$ be the maximum expected value of the n-1 stage return and let it be a function that is defined for all $X_{n-1}' \in S_{X_{n-1}'}$, $1 \leq j \leq I$, $2 \leq n \leq N$. Note that $f_{n-1,j}(X_{n-1}')$ can be visualized as the result of having maximized some objective function relating to the n-1 stage structure subject to the condition that the Markov state variable takes the value j and the residual state vector takes the value X_{n-1}' at stage n-1.

The return function $g_{nij}[X_n', D_n, f_{n-1,j}(X_{n-1}')]$ characterizes the composition of the return from stage n with the maximum expected value of the return from the remaining n-1 stages. Suppose that for all $X_n' \in S_{X_n'}, D_n \in S_{D_n}(X_n')$, $1 \leq i \leq I$, $1 \leq j \leq I$, $2 \leq n \leq N$, the return function $g_{nij}[\cdots]$ has the following two properties:

1. monotonicity: If $f_{n-1,j}(X_{n-1}')$ is replaced by an independent variable, say y, $g_{nij}[X_n', D_n, y]$ is defined and it is a monotonically nondecreasing function of y.

2. equivalence: If Z is a random variable and E(Z) represents the expected value of Z, then

$$E\left[g_{nij}(X_n',D_n,Z)\right] = g_{nij}\left[X_n',D_n,E(Z)\right]$$

Given these properties, the function $g_{nij}\left[X_n',D_n,f_{n-1,j}(X_{n-1}')\right]$ represents the maximum expected value of return that is obtainable from the n stage system for given values of X_n',D_n,i,j.[3] It follows that the functional equation

$$f_{ni}(X') = \underset{D_n \in S_{D_n}(X_n')}{\text{Max}} \sum_{j=1}^{I} p_{ij}(D_n)\, g_{nij}\left[X_n',D_n,f_{n-1,j}(t_{nij}(X_n',D_n))\right] \tag{18}$$

represents the maximum expected value of the n stage return subject to the condition that the residual state vector takes the value X_n' and the Markov state variable takes the value i at stage n. The function $f_{ni}(X_n')$ is defined for all $X_n' \in S_{X_n'}$, $1 \leq i \leq I$, $2 \leq n \leq N$.

Thus, $f_{1i}(X_1')$ is the maximum expected value of the one stage return and we have shown that if $f_{n-1,j}(X_{n-1}')$ is the maximum expected value of the n-1 stage return, then $f_{ni}(X_n')$ is the maximum expected value of the n stage return subject to suitable conditions. So by mathematical induction, $f_{ni}(X_n')$ is the maximum expected value of the n stage return for all $X_n' \in S_{X_n'}$, $1 \leq i \leq I$, $1 \leq n \leq N$.

The sufficient conditions for the optimality of $f_{ni}(X_n')$ will now be summarized for the special Markovian decision process. For completeness, let

$$g_{1ij}\left[X_1',D_1,f_{oj}(X_o')\right] \equiv g_{1ij}(X_1',D_1) \tag{19}$$

for all X_1',D_1,i,j in their respective sets.

[3] A proof of this statement appears in Appendix B.

If we are given the transformation relationships

$$X'_{n-1} = t_{nij}(X'_n, D_n) \; ; \qquad 2 \leq n \leq N \qquad (20)$$
$$1 \leq i \leq I$$
$$1 \leq j \leq I$$

there must exist sets of allowable values of X'_n and D_n, $S_{X'_n}$ and $S_{D_n}(X'_n)$, such that if $X'_n \in S_{X'_n}$ and $D_n \in S_{D_n}(X'_n)$, then $t_{nij}(X'_n, D_n) \in S_{X'_{n-1}}$ for all $1 \leq i \leq I$, $1 \leq j \leq I$. Also, the function $g_{1ij}\left[X'_1, D_1, f_o(X_o)\right]$ must be defined for all $X'_1 \in S_{X'_1}, D_1 \in S_{D_1}(X'_1)$, $1 \leq i \leq I$ and $1 \leq j \leq I$. Furthermore, the function $g_{nij}\left[X'_n, D_n, f_{n-1,j}(X'_{n-1})\right]$ must possess the monotonicity and equivalence properties that were defined previously in this section for all $X'_n \in S_{X'_n}$, $D_n \in S_{D_n}(X'_n)$, $1 \leq i \leq I$, $1 \leq j \leq I$, $2 \leq n \leq N$.

If the foregoing conditions are met, then recursive application of equation (18) will yield the maximum expected value of the n stage return for all $X'_n \in S_{X'_n}$, $1 \leq i \leq I$, $1 \leq n \leq N$.

A Markov Decision Process with Unobservable Markov State Transitions

The discussion so far has presupposed a knowledge of the state of the system at all stages. It will be useful for later reference to indicate a functional equation for a case where the Markov state transitions are not observable, hence the Markov state of the system is only probabilistically known for $n < N$.

Let $\pi_N(i)$ be the probability function of the Markov state at stage N. Let $\pi_n(i; \pi_N, D_N, D_{N-1}, \cdots, D_{n+1})$ be the probability function

of i at stage n. The function $\pi_n(i;\ \pi_N, D_N,\ D_{N-1}, ---, D_{n+1})$ can be evaluated by multiplication of $\pi_N(i)$ and the succeeding matrices of transition probabilities.

The functional equation for this situation can be developed by starting from equation (18). Since the Markov state, i, is only known probabilistically at each stage, the maximum n stage return will be taken as the maximum expected value of the return taken over the random variable i. This maximum n stage return depends on the initial probability function π_N and it depends on the decisions made at stages N, N-1,---,n+1; it will be denoted $f_n(X_n',\ \pi_N,\ D_N, --- D_{n+1})$. The return function may depend on the stage and the Markov states involved; hence, the return function is defined as $g_{nij}\left[X_n', D_n, f_{n-1}(X_{n-1}', \pi_N, D_N, ---, D_n)\right]$. The quantity $\left\{\ \cdot\ \right\}$ in equation (18) thus becomes

$$\left\{ \sum_{j=1}^{I} p_{ij}(D_n)\ g_{nij}\left[X_n', D_n, f_{n-1}(X_{n-1}', \pi_N, D_N, ---, D_n)\right] \right\}$$

Since this quantity is the maximum expected value of the n stage return for given n, X_n', D_n, and i and since i is a random variable, the maximum expected value of the n stage return for given n, X_n', and D_n is given by

$$\sum_{i=1}^{I} \pi_n\ (i;\ \pi_N,\ D_N,\ D_{N-1}, ---, D_{n+1}) \left\{\ \cdot\ \right\}$$

Taking the maximum over D_n gives

$$f_n(X_n',\ \pi_N,\ D_N,\ D_{N-1}, ---, D_{n+1}) = \underset{D_n \in S_{D_n}(X_n')}{\mathrm{Max}} \sum_{i=1}^{I} \pi_n(i; \pi_N, D_N, D_{N-1}, --- D_{n+1})$$

$$\sum_{j=1}^{I} p_{ij}(D_n)\ g_{nij}\left[X_n',\ D_n,\ f_{n-1}(X_{n-1}',\ \pi_N,\ D_N,\ D_{N-1}, ---, D_n)\right] \qquad (21)$$

The high dimensionality of equation (21) will probably render it impractical to implement for most cases, but it is useful to have an indication of the type of relationship that is involved.

If an implementation of equation (21) were undertaken, it would involve determining $\pi_n(i; \pi, D_N, ---, D_{n+1})$ for all feasible sequences of D_n and for all n. The results could then be used in equation (21) to determine the optimum decision at stage n, $D_n^*(X_n', \pi_N, D_N, D_{N-1}, ---, D_{n+1})$. The residual state variable transformation relations could then be used to determine the optimum sequence $(D_N^*, ---D_1^*)$.

CHAPTER III

THE SIMPLE E_H DUEL[1]

Recursive Relationships

Having developed some basic functional equations, we will now consider how they might be usefully applied to the air-to-ground attack problem. The number of hits achieved by the aircraft is a random variable. This chapter considers situations where the return of interest is the expected value of the number of hits. The cost is in terms of the number of aircraft lost.

The first problem will be a very simple duel in which we seek to maximize the expected value of the number of hits (hereafter referred to as "expected hits") achievable by a single aircraft attacking a defended ground target. This will be referred to as the "simple E_H duel."

Suppose an aircraft with x_N bombs on board is to make not more than N weapon delivery passes. A salvo of d_n bombs is to be delivered on the pass n subject to the restrictions that

$$\sum_{n=1}^{N} d_n \leq x_N \tag{1}$$

$$d_n \geq 0 \qquad\qquad 1 \leq n \leq N \tag{2}$$

[1]Throughout this work, a duel involving allocation of weapons among passes and not including probabilistic target acquisition or multiple modes of attack is referred to as a "simple" duel.

Throughout this work, x_n and d_n are assumed to be discrete variables. The unit of weapons will normally be referred to as a bomb; however, it could represent a cluster of bombs or a salvo of rockets.

Let S_T represent the probability that on a given pass, the aircraft survives long enough to deliver weapons. Let S_u represent the conditional probability that the aircraft survives the pass given that it survives long enough to deliver weapons on that pass. In this context, we will interpret "not surviving" as the occurrence of an enemy action that prevents the aircraft from participating further in the attack. This could mean anything from immediate kill of the aircraft to relatively minor damage.

To apply recursive analysis to this problem, a pass will be identified as one stage. It is not necessary, but it will be convenient to number the passes in reverse order so that the chronologically last pass is pass number one. The state of the system when preparing for pass n, i.e., at stage n, can be completely defined by specifying the number of weapons remaining, denoted x_n. Thus x_n will be the state variable playing the role that the state vector, X_n, played in the general formulation of Chapter II. Let the number of weapons delivered on pass n, d_n, be the decision variable analogous to D_n.

If d_n weapons are delivered on pass n, then

$$x_{n-1} = x_n - d_n \quad ; \qquad 1 \leq n \leq N \tag{3}$$

Thus, since the state variable is subject to deterministic transformation, we have a deterministic decision process. In studying this process, the stage return, $r_H(d_n)$, will be the expected hits per salvo

as a function of salvo size. The function $f_n(x_n)$ will represent the maximum expected hits achievable on the remaining n passes if there are x_n weapons remaining.

To satisfy the sufficient conditions for optimality, we must show the existence of sets S_{x_n} and $S_{d_n}(x_n)$ such that $x_n \in S_{x_n}$ and $d_n \in S_{d_n}$ for all $1 \leq n \leq N$. It can be seen by inspection of the transformation relationship, equation (1), and considering the non-negativity of d_n, that the sets

$$S_{x_n} = \left\{x_n : x_n \in \left\{0, 1, ---, \bar{x}_N\right\}\right\} \quad ; \quad 1 \leq n \leq N \qquad (4)$$

and

$$S_{d_n}(x_n) = \left\{d_n : d_n \in \left\{0, 1, ---, x_n\right\}\right\} \quad ; \quad 1 \leq n \leq N \qquad (5)$$

satisfy the requirement. The quantity $\bar{x}_N > 0$ is the largest number of weapons that might be of interest at pass N.[2]

The return function must be formulated next. If one pass remains and d_1 weapons are to be delivered, then the expected hits is the probability of surviving to the point of weapon delivery times $r_H(d_1)$; leftover weapons have no value. Thus,

$$g_1 \left[x_1, d_1, f_o(x_o)\right] = S_T \, r_H(d_1) \quad ; \quad n = 1 \qquad (6)$$

where the functions in equation (6) must be defined for all $x_1 \in S_{x_1}$ and $d_1 \in S_{d_1}(x_1)$.

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[2] Once a value of $\bar{x}_N$ is established, the actual number of weapons available at stage N, x_N, can take any value such that $x_N \leq \bar{x}_N$. The value of $\bar{x}_N$ only establishes the range of values of x_N for which solutions will be obtained.

When more than one pass remains, the composition of the maximum n-1 stage return, $f_{n-1}(x_{n-1})$, with the stage n return, $r_H(d_n)$, can be accomplished as follows. The expected hits for pass n is the probability of surviving to the point of weapon release times the expected hits obtained on the salvo that is delivered. The maximum expected hits for the remaining n-1 passes is the probability of surviving pass r times $f_{n-1}(x_{n-1})$. Thus,

$$g_n \left[x_n, d_n, f_{n-1}(x_{n-1})\right] = S_T\, r_H(d_n) + S_T\, S_u\, f_{n-1}(x_{n-1}) \quad ; \quad 2 \le n \le N \tag{7}$$

By examining equation (6), we see that if $r_H(d_1)$ is defined for all $x_1 \in S_{x_1}$ and $d_1 \in S_{d_1}(x_1)$, then the return function for n = 1 is defined for all $x_1 \in S_{x_1}$ and $d_1 \in S_{d_1}(x_1)$.

Furthermore, when $2 \le n \le N$, we see from equation (7) that if $r_H(d_n)$ is defined for all $x_n \in S_{x_n}$ and $d_n \in S_{d_n}(x_n)$, then the return function meets the monotonicity requirement that is defined in Chapter II for a deterministic decision process. The previous statement is true because $S_T \ge 0$ and $S_u \ge 0$.

Note that the function $g_n\left[\cdots\right]$ is indexed by stage so its form and its associated coefficients can therefore be made stage dependent. Accordingly, the function r_H and the quantities S_T and S_u could carry the subscript n. This will be understood to be true throughout this work but the subscript is not carried explicitly because it would make the notation more awkward without adding significantly to the content of the work.

We have now defined sets and return functions so that the sufficient conditions for optimality are satisfied. Recursive application of equation (II-5) will yield the desired $f_n(x_n)$.[3] Substituting equations (5) and (6) into (II-5) and considering equation (4) gives

$$f_1(x_1) = \max_{0 \leq d_1 \leq x_1} \left[S_T\, r_H(d_1) \right] \quad ; \quad 0 \leq x_1 \leq \bar{x}_N \tag{8}$$

Since $r_H(d)$ is assumed to be a nondecreasing function of d,

$$f_1(x_1) = S_T\, r_H(x_1) \quad ; \quad 0 \leq x_1 \leq \bar{x}_N \tag{9}$$

Substitution of equations (3), (5), and (7) into (II-5) and considering equation (4) gives

$$f_n(x_n) = \max_{0 \leq d_n \leq x_n} \left[S_T\, r_H(d_n) + S_T\, S_u\, f_{n-1}(x_n - d_n) \right] \; ; \; \begin{matrix} 0 \leq x_n \leq \bar{x}_N \\ 2 \leq n \leq N \end{matrix} \tag{10}$$

Equations (9) and (10) constitute the recursive relationships that apply to the simple duel.

Concavity of $f_n(x_n)$

It is generally a supportable assumption that $r_H(d_n)$ is a concave function of d_n. It can be shown that when this is true, the $f_n(x_n)$ determined from equations (9) and (10) are concave functions of x_n for all

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[3]The equation numbers start from (1) in each chapter. When referring to an equation of the current chapter, its arabic number will be used. When referring to an equation of a previous chapter, its arabic number will be preceded by the appropriate Roman numeral chapter number.

$1 \leq n \leq N$. This can be proved as follows. First, if $n = 1$, equation (9) shows that since $S_T \geq 0$ and $r_H(d_1)$ is concave in d_1, then $f_1(x_1)$ is concave in x_1.

Now if $2 \leq n \leq N$, consider equation (10). Note that if $r_H(d_n)$ is concave and if $f_{n-1}(x_n - d_n)$ is a concave function of $x_n - d_n$, then since $S_T \geq 0$ and $S_u \geq 0$, the expression

$$g'(x_n, d_n) = S_T\left[r_H(d_n) + S_u f_{n-1}(x_n - d_n)\right] \tag{11}$$

is a concave function of x_n and d_n. To show this for a given n let u_1 and u_2 be arbitrary values of x_n and let v_1 and v_2 be arbitrary values of d_n where $u_1, u_2 \in S_{x_n}$ and $v_1, v_2 \in S_{d_n}(x_n)$. For $0 \leq \lambda \leq 1$,

$$\begin{aligned}
g'(x_n,d_n) &= S_T\left[r_H(\lambda v_1+(1-\lambda)v_2) + S_u f_{n-1}(\lambda u_1+(1-\lambda)u_2-\lambda v_1-(1-\lambda)v_2)\right] \\
&= S_T\left\{r_H(\lambda v_1+(1-\lambda)v_2) + S_u f_{n-1}\left[\lambda(u_1-v_1) + (1-\lambda)(u_2-v_2)\right]\right\} \\
&\geq S_T\left\{\lambda r_H(v_1) + (1-\lambda) r_H(v_2) + S_u\left[\lambda f_{n-1}(u_1-v_1) + (1-\lambda)f_{n-1}(u_2-v_2)\right]\right\} \\
&= S_T\left\{\lambda\left[r_H(v_1) + S_u f_{n-1}(u_1-v_1)\right] + (1-\lambda)\left[r_H(v_2) + S_u f_{n-1}(u_2-v_2)\right]\right\} \\
&= \lambda g'(u_1,v_1) + (1-\lambda) g'(u_2,v_2)
\end{aligned} \tag{12}$$

A proof given by Bellman (5, p. 21) can now be used to show that the following is a concave function of x_n.[4]

$$f(x) = \max_{0 \leq d_n \leq x_n} g'(x_n,d_n) \tag{13}$$

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[4]This proof by Bellman is included here rather than simply referencing it because it is vital to include some such proof in the demonstration of concavity. The reader may not have ready access to Ref. 5 and furthermore, some small notational adjustments have been made in adapting the proof for our purpose.

For a given n and using the previous definitions of u_1, u_2, v_1, v_2, λ,

$$f_n(x_n) = f_n(\lambda u_1 + (1-\lambda)\, u_2) = \underset{0 \le d_n \le \lambda u_1+(1-\lambda)u_2}{\mathrm{Max}} g'(\lambda u_1+(1-\lambda)u_2, d_n) \tag{14}$$

The quantity d_n can be replaced by $d_n = \lambda v_1 + (1-\lambda)v_2$ where v_1 and v_2 range independently over the intervals $0 \le v_1 \le u_1$, $0 \le v_2 \le u_2$. Now for given λ, we seek optimizing values of v_1 and v_2.

$$f_n(\lambda u_1+(1-\lambda)u_2) = \underset{\substack{0 \le v_1 \le u_1 \\ 0 \le v_2 \le u_2}}{\mathrm{Max}} g'\left[\lambda u_1+(1-\lambda)u_2, \lambda v_1+(1-\lambda)v_2\right] \tag{15}$$

Since $g'(x_n, d_n)$ is concave in x_n, d_n,

$$g'\left[\lambda u_1+(1-\lambda)u_2,\ \lambda v_1+(1-\lambda)v_2\right] \ge \lambda g'(u_1,v_1) + (1-\lambda)\ g'(u_2,v_2) \tag{16}$$

Hence,

$$f_n(\lambda u_1+(1-\lambda)u_2) \ge \underset{\substack{0 \le v_1 \le u_1 \\ 0 \le v_2 \le u_2}}{\mathrm{Max}} \left[\lambda g'\ (u_1,v_1) + (1-\lambda)\ g'\ (u_2,v_2)\right]$$

$$\ge \lambda \underset{0 \le v_1 \le u_1}{\mathrm{Max}} g'(u_1,v_1) + (1-\lambda) \underset{0 \le v_2 \le u_2}{\mathrm{Max}} g'(u_2,v_2)$$

$$\ge \lambda f_n(u_1) + (1-\lambda)\ f_n(u_2) \tag{17}$$

The foregoing shows that if $r_H(d_n)$ is concave in d_n, then $f_1(x_1)$ is concave in x_1. It further shows that if in addition $f_{n-1}(x_n - d_n)$ is concave in $x_n - d_n$, then $f_n(x_n)$ is concave in x_n. Thus, by

mathematical induction, $f_n(x_n)$ as defined by equations (9) and (10) is concave in x_n for all $1 \leq n \leq N$.

This concavity is a useful property because it simplifies the determination of the optimum salvo size, $d_n^*(x_n)$ when applying equation (10). This simplification comes about in finding the maximum because, given concavity, any local maximum is also a global maximum.

Numerical Example

As a numerical example to illustrate the application of equations (9) and (10), let

$$N = 6 \text{ passes}$$

$$\bar{x}_N = 8 \text{ weapons}$$

$$S_T = 1.0$$

$$S_u = 0.98$$

and suppose $r_H(d_n)$ has the form

$$r_H(d_n) = \Psi\ (1-\theta^{d_n}) \quad ; \quad 0 \leq \theta \leq 1.0 \qquad (18)$$

$$\geq 0$$

For this example, let

$$\Psi = 1.0$$

$$\theta = 0.84$$

Throughout this work, considerable use will be made of this form, i.e., equation (18), for the salvo effectiveness function. Note that in equation (18), $r_H(d_n)$ is a monotonically nondecreasing concave function of d_n. This functional form provides a two parameter family of functions that can be used to approximate a considerable variety of possible salvo effectiveness functions. The form is appropriate

because salvo effectiveness functions are monotonically nondecreasing functions and they tend to be concave, i.e., salvo effectiveness increases monotonically with salvo size, but there tends to be a diminishing return with increased salvo size. Furthermore, this form will accommodate the case where we seek to kill the target and individual members of the salvo are assumed to be delivered independently. For this latter case, θ is interpreted as the probability that the target survives a single weapon and $\Psi = 1.0$.

Figure 3 illustrates the $r_H(d_n)$ function and Table 1 shows the results of carrying out the calculations. From this table, we can read the optimum allocation of weapons among the passes and the maximum expected hits for any initial bomb load up to eight and for any limiting number of passes up to six. Given eight bombs ($x_N = 8$) and a maximum of six passes ($N = 6$), then the expected hits, $f_6(8)$, is 1.179. The optimum allocation of weapons is $(d_6^*, d_5^*, d_4^*, d_3^*, d_2^*, d_1^*) = (2,2,1,1,1,1)$. It is satisfying to note that for any given n, $f_n(x_n)$ is an increasing function of x_n and for any given x_n, $f_n(x_n)$ is an increasing function of n as would be expected.

It is interesting to note that fifty-three seconds were required for a Fortran IV program on an IBM 7094 to generate 120 such tables with $N = 10$ and $\bar{x}_N = 12$. The concavity property was not used in this program, i.e., complete enumeration was carried out. Accordingly, the running time could probably be reduced by using the concavity property.

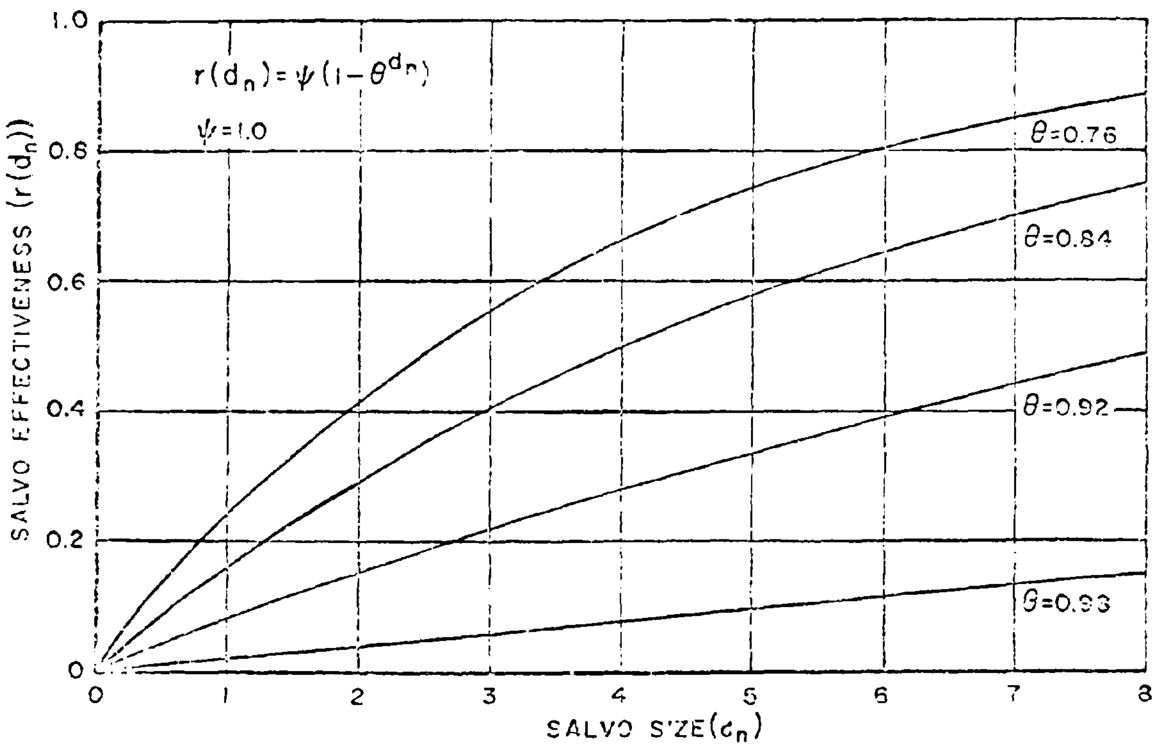


Fig. 3.--Salvo Effectiveness Versus Salvo Size.

TABLE 1

EXAMPLE RESULTS FOR THE SIMPLE E_H DUEL

$\Psi = 1.0$; $\theta = 0.84$; $S_T = 1.0$; $S_u = 0.98$; $N = 6$; $\bar{x}_N = 8$

x_1	d_1^*	$f_1(x_1)$	x_2	d_2^*	$f_2(x_2)$	x_3	d_3^*	$f_3(x_3)$	x_4	d_4^*	$f_4(x_4)$	x_5	d_5^*	$f_5(x_5)$	x_6	d_6^*	$f_6(x_6)$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	.160	1	1	.160	1	1	.160	1	1	.160	1	1	.160	1	1	.160
2	2	.294	2	1	.317	2	1	.317	2	1	.317	2	1	.317	2	1	.317
3	3	.407	3	2	.451	3	1	.470	3	1	.470	3	1	.470	3	1	.470
4	4	.502	4	2	.583	4	2	.605	4	1	.621	4	1	.621	4	1	.621
5	5	.582	5	3	.696	5	2	.737	5	2	.755	5	1	.769	5	1	.769
6	6	.649	6	3	.806	6	2	.866	6	2	.887	6	2	.903	6	1	.913
7	7	.705	7	4	.901	7	3	.979	7	2	1.016	7	2	1.035	7	2	1.048
8	8	.752	8	4	.994	8	3	1.089	8	2	1.143	8	2	1.164	8	2	1.179

Non Recursive Form

Insight can be gained by stating the problem in non recursive form. Let $R_n(d_n, d_{n-1},---,d_1)$ denote the expected hits obtainable on passes n,---,1 for a given allocation. The constraints stated in equations (1), (2), and (3) must be met. With one pass remaining, we have

$$R_1(d_1) = S_T \; r_H(d_1) \tag{19}$$

With two passes remaining,

$$\begin{aligned} R_2(d_2, d_1) &= S_T r_H(d_2) + S_T \; S_u \; R_1(d_1) \\ &= S_T r_H(d_2) + S_T^2 \; S_u \; r_H(d_1) \end{aligned} \tag{20}$$

With three passes remaining,

$$\begin{aligned} R_3(d_3, d_2, d_1) &= S_T \; r_H(d_3) + S_T \; S_u \; R_2(d_2, d_1) \\ &= S_T \; r_H(d_3) + S_T^2 \; S_u \; r_H(d_2) + S_T^3 \; S_u^2 \; r_H(d_1) \end{aligned} \tag{21}$$

With N passes remaining,

$$R_N(d_N,---,d_1) = S_T \sum_{n=1}^{N} (S_T \; S_u)^{N-n} \; r_H(d_n) \tag{22}$$

Thus, the original optimization problem might be stated as follows:

$$f_N(x_N) = \max_{d_N,---,d_1} S_T \sum_{n=1}^{N} (S_T \; S_u)^{N-n} \; r_H(d_n) \tag{23}$$

subject to

$$\sum_{n=1}^{N} d_n \leq x_N \tag{24}$$

$$d_n \geq 0 \quad ; \quad 1 \leq n \leq N \tag{25}$$

This problem has the same form as a problem of sequential allocation with discounting. The product $S_T\ S_u$ might be interpreted as a discount factor as used, for example, by Howard (17).

A Parametric Investigation

We have seen an example of the type of solution that is obtained for given values of the input parameters. It is of interest to see how the optimal allocation, $(d_N^*, d_{N-1}^*, \text{---}, d_1^*)$, and the maximum expected hits, $f_N(x_N)$, vary for a range of input parameter values. This investigation can be facilitated by making use of the non recursive statement of the problem, equations (23), (24), and (25).

Assume that $r_H(d_n)$ can be expressed by equation (18). Then equation (23) becomes

$$f_N(x_N) = S_T \Psi \left[\underset{d_1, \text{---}, d_N}{\text{Max}} \sum_{n=1}^{N} (S_T\ S_u)^{N-n} (1-\theta^{d_n}) \right] \tag{26}$$

With the problem expressed in this form, it is clear that the optimal allocation for given N and x_N depends only on the two quantities $(S_T S_u)$ and θ and not on Ψ.

Table 2 shows the variation of $(d_N^*, d_{N-1}^*, \text{---}, d_1^*)$ versus $(S_T\ S_u)$ and θ when $x_N = 8$ and $N = 8$. The appropriate salvo effectiveness functions are shown in Figure 3. Note that because of the

TABLE 2

OPTIMUM WEAPON ALLOCATION FOR A SIMPLE E_H DUEL

$N = 8 \quad x_N = 8$

Probability of the aircraft surviving one pass $(S_T S_u)$	.76	.84	.92	.98
1.0	1,1,1,1,1,1,1,1	1,1,1,1,1,1,1,1	1,1,1,1,1,1,1,1	1,1,1,1,1,1,1,1
.98	1,1,1,1,1,1,1,1	1,1,1,1,1,1,1,1	2,2,1,1,1,1	4,3,1
.92	2,2,1,1,1,1	2,2,2,1,1	3,3,2	6,2
.86	3,2,2,1	3,2,2,1	4,3,1	8
.80	3,2,2,1	4,3,1	5,3	8

Value of the Salvo Effectiveness Function Parameter θ

Each block contains the non zero elements of $(d_N^*,\text{---},d_1^*)$.

discounting structure, if $n_1 > n_2$, then $d_{n_1}^* \geq d_{n_2}^*$. Thus with $x_N = 8$, there will never be any reason to make more than eight passes so all the $(d_N^*,---,d_1^*)$ in Table 2 are optimum.

As might be expected for given θ, higher survival probability, i.e., larger $(S_T\ S_u)$, leads to more passes. Also, for given $(S_T\ S_u)$, smaller values of θ lead to more passes. This is reasonable if we view θ as controlling the rate at which $r_H(d_n)$ approaches its asymptotic level Ψ ; see equation (18) and Figure 3. If θ is small, it means that the marginal value of increasing d_n decreases rapidly as d_n increases. This makes small salvo size and a corresponding larger number of passes more advantageous at a fixed survival level.

It is often true that the probability of survival per pass, $(S_T\ S_u)$, is greater than 0.98. If it were not, a sustained air-to-ground effort would probably be impractical. This being the case, unless θ is close to 1.0, Table 2 indicates that the optimal allocation tends to be an even distribution of weapons among passes. The case where both $(S_T\ S_u)$ and θ approach 1.0 enters a region where the optimal allocation is quite sensitive to both $(S_T\ S_u)$ and θ as is illustrated by Table 2 but is rather uninteresting otherwise as is borne out by Figure 4.

Figure 4 shows the variation of $f_8(8)$ versus S and θ when $S_T = \Psi = 1.0$. This figure gives the impression that as the value of θ decreases, $f_8(8)$ becomes more sensitive to the value of S_u. This is reasonable because as θ decreases, the tendency is to make more passes, thus making survival more important. Based on the same type of reasoning, it is reasonable for $f_8(8)$ to be more sensitive to S_u as the value of S_u approaches 1.0.

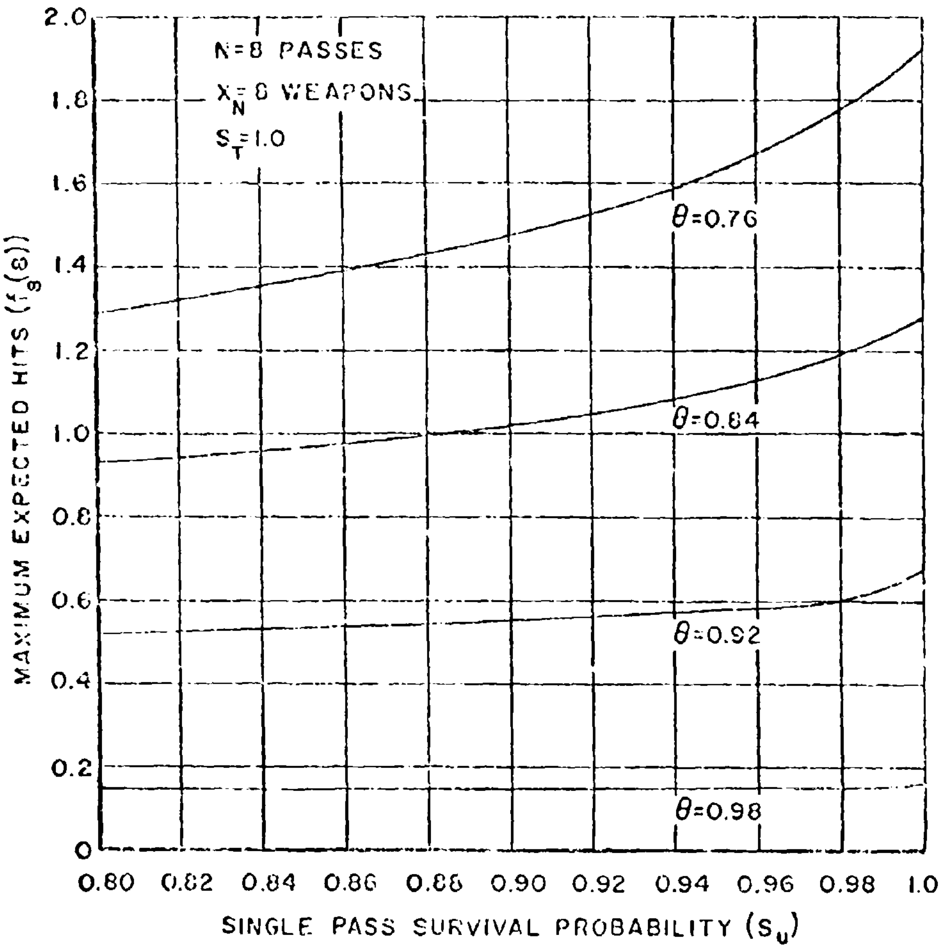


Fig. 4.--Maximum return versus survival probability and weapon effectiveness.

Return Versus Attrition

A very important concept will now be introduced. The discussion so far has considered maximizing the expected hits per duel. This has been referred to as the return. The example of Table 1 shows that the return is appreciably higher for six passes than it is for only one pass.

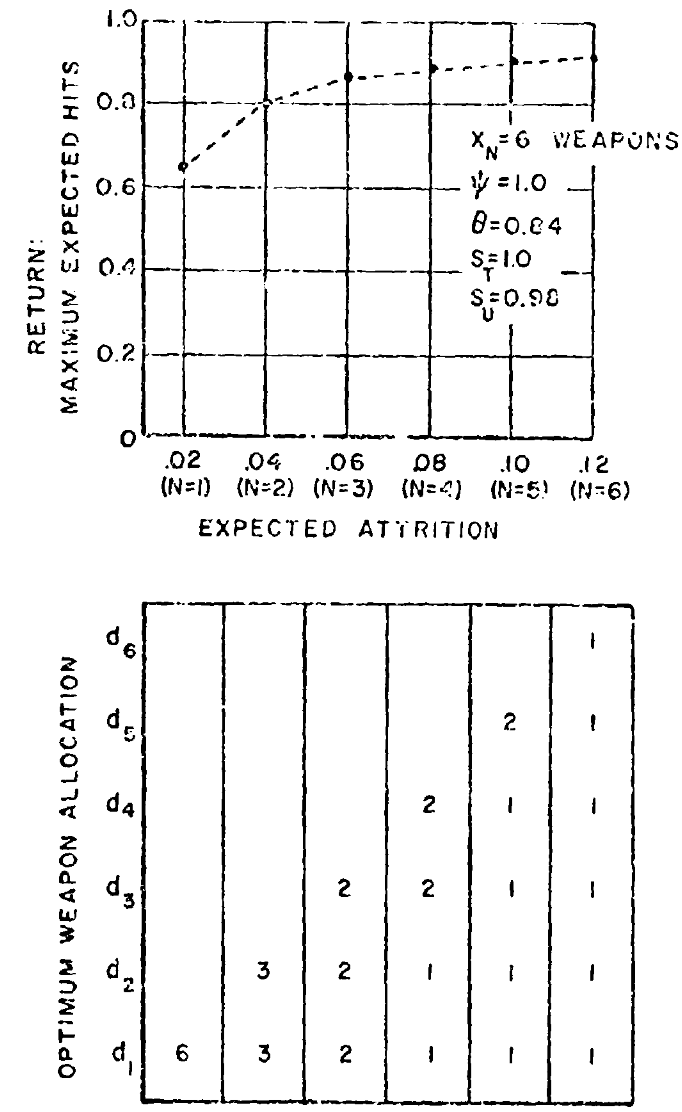
The foregoing tells only part of the story. In addition to the return, the cost must be considered. Cost will be measured in terms of aircraft lost or "attrition." When only one aircraft is involved as in the present discussion, expected attrition will indicate the probability of the aircraft not surviving.

Note that for a complete model, the cost should also include other factors such as the cost of weapons and the cost of fuel. These will be excluded here since they are often negligible compared to the cost of replacing aircraft and pilots. Further, including such other costs is generally a straightforward procedure. For the simple duel, the maximum expected attrition is given by

$$L_D = 1 - (S_T\ S_u)^N \tag{28}$$

Note that the actual expected attrition may be different from the maximum expected attrition since the optimum solution may call for less than N passes.

Figure 5 shows a return-versus-attrition function for the example of Table 1. This figure applies to the case where six weapons are available at stage N, i.e., $x_N = 6$, and N is varied from 1 to 6.



	.02 (N=1)	.04 (N=2)	.06 (N=3)	.08 (N=4)	.10 (N=5)	.12 (N=6)
d_6						1
d_5					2	1
d_4				2	1	1
d_3			2	2	1	1
d_2		3	2	1	1	1
d_1	6	3	2	1	1	1

Fig. 5.--Return-versus-attrition function and corresponding optimum weapon allocation for the simple E_H duel.

This figure shows the return as a function of the actual expected attrition because in the example, it is optimal to make N passes if $N \leq 6$.

The expected attrition entries from left to right on the abscissa scale in Figure 5 are the values obtained from equation (28) when $S_T\ S_u = 0.98$ and $N = 1,2,3,4,5,6$, respectively. At each level of expected attrition, the return is the maximum expected hits assuming that an optimum weapon allocation is used. For example, the attrition level of 0.06 results when $N = 3$. We see from Table 1 that when $N = 3$ and $x_3 = 6$, the maximum expected hits, $f_3(6)$ is 0.866. Thus, the ordinate value is 0.866 corresponding to an expected attrition of 0.06 in Figure 5. The optimum weapon allocation when $N = 3$ and $x_3 = 6$ can also be read from Table 1 as $(d_3^*, d_2^*, d_1^*) = (2,2,2)$.

We have now developed a model for optimizing the simple E_H duel and shown how its results can be used to determine the return-versus-attrition function. We will proceed to complicate and modify our notion of a duel, but the return-versus-attrition function will be a basic characterizing feature of all the duels that will be studied.

In addition to studying other duels, determining which of the points on the return-versus-attrition function corresponds to the most desirable weapon allocation will also be the subject of considerable subsequent discussion.

CHAPTER IV

GENERALIZING THE E_H DUEL[1]

Incorporating Probabilistic Target Acquisition

We continue considering a multiple pass air-to-ground attack where the return is in terms of expected hits and the cost is in terms of the number of aircraft lost. All the features of the simple E_H duel are still present and some additional factors will be included.

The simple E_H duel assumes the weapons allocated to a pass are sure to be delivered if the aircraft survives long enough. In present close support operations, the presence of a forward air controller who directs the attack generally makes this a valid assumption. Likewise, the assumption is reasonable for many interdiction attacks on easily acquired targets such as bridges or harbor facilities.

There are also interesting situations, however, where the target is not easily acquired and there is no forward air controller. This occurs under conditions which may include night, bad weather, and obscure targets. The ability of the aircraft to acquire targets is related to its sensor capability and its navigation system. Considerations of this sort are becoming increasingly important in the analysis of weapon systems that are technologically advanced.

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[1]It may be useful to the reader to refer to Appendix A either in conjunction with or after reading Chapter IV. That appendix indicates how the problem of Chapter IV can be approached by first making a non recursive statement of the problem and then developing the recursive relationships in a manner similar to that of Nemhauser (22).

Consider the following abstraction of the target acquisition aspect of the problem. This discussion extends the general discussion that appeared in the second section of Chapter I. An "initial" pass is made when the target has not been acquired previously on the same sortie. Once target acquisition has occurred, additional passes on the same sortie will be referred to as "subsequent" passes. This distinction allows for improved target acquisition capability once the target has been seen.

When an initial pass is made, one of the following three events will occur:

A_0^* : the target is not acquired

A_0D^*: the target is acquired but weapon delivery is not possible

A_0D: the target is acquired and weapon delivery is possible.

The corresponding events for a subsequent pass are symbolized A_1^*, A_1D^*, and A_1D. Since weapon delivery may or may not be possible on pass n, transformation of the state variable x_n is probabilistic and we have a Markovian decision process.

To apply the recursive relationship that was developed in Chapter II for a Markovian decision process, equation (II-18), three levels for the Markov state variable i are defined as follows:

$i = 1$: acquisition has not yet occurred

$i = 2$: acquisition has occurred and no weapons were delivered on the most recent pass

$i = 3$: acquisition has occurred and weapons were delivered on the most recent pass.

In the present case, the transition probability is independent of the stage n decision, d_n, so we shall simply denote the transition probability as p_{ij}. The matrix of transition probabilities is given in Figure 6 for the system that has been described.

	j = 1 A*	j = 2 AD*	j = 3 AD
i = 1:A*	$p_{11} = 1 - P(A_o)$	$p_{12} = P(A_oD^*)$ $= P(A_o) - P(A_oD)$	$p_{13} = P(A_oD)$
i = 2:AD*	$p_{21} = 0$	$p_{22} = 1 - P(A_1D)$	$p_{23} = P(A_1D)$
i = 3:AD	$p_{31} = 0$	$p_{32} = 1 - P(A_1D)$	$p_{33} = P(A_1D)$

Fig. 6.--Markov state transition probabilities.

Transformation of the state variable x_n depends on the transition that occurs in the Markov state (analogous to equation (II-9)). If transition is to Markov state 1 or 2, no weapons are delivered. If transition is to Markov state 3, d_n weapons are delivered, hence for all $1 \leq n \leq N$ and $1 \leq i \leq 3$,

$$\begin{aligned} x_{n-1} &= x_n \quad ; \quad j = 1,2 \\ x_{n-1} &= x_n - d_n \quad ; \quad j = 3 \end{aligned} \tag{1}$$

It can be seen by inspection of the transformation relation equation (1) and considering the non negativity of d_n that the following

sets have the required property that $x_n \in S_{x_n}$ and $d_n \in S_{d_n}(x_n)$ for all $1 \le n \le N$, $1 \le i \le 3$, $1 \le j \le 3$.

$$S_{x_n} = \left\{ x_n : x_n \in \left\{ 0, ---, \bar{x}_N \right\} \right\} \tag{2}$$

$$S_{d_n}(x_n) = \left\{ d_n : d_n \in \left\{ 0, ---, x_n \right\} \right\} \tag{3}$$

Return functions that have the required monotonicity and equivalence as defined in Chapter II will now be developed. Weapon delivery occurs only when transition is to Markov state 3 and the aircraft survives to the point of weapon release. Thus, if $n = 1$ and for all $1 \le i \le 3$,

$$\begin{aligned} g_{1ij}\left[x_1, d_1, f_{oj}(x_o)\right] &= 0 \qquad ; \; j = 1,2 \\ &= S_T \, r_H(d_1) \qquad ; \; j = 3 \end{aligned} \tag{4}$$

When transition is to Markov states 1 or 2, no weapons are delivered. If $f_{ni}(x_n)$ represents the maximum expected hits achievable in n passes with x_n weapons and in Markov state i, then the return associated with transition to Markov states 1 or 2 is the probability of surviving pass n times $f_{n-1,j}(x_{n-1})$. The return associated with transition to Markov state 3 is the probability of surviving to the point of weapon delivery times the expected hits achievable by the salvo that is delivered on pass n plus the probability of surviving pass n times $f_{n-1,3}(x_{n-1})$. Thus, if $n > 1$ and for all $1 \le i \le 3$

$$\begin{aligned} g_{nij}\left[x_n, d_n, f_{n-1,j}(x_{n-1})\right] &= S_T \, S_u \, f_{n-1,j}(x_{n-1}) \qquad ; \; j = 1,2 \\ &= S_T \, r_H(d_n) + S_T \, S_u \, f_{n-1,3}(x_{n-1}) \qquad ; \; j = 3 \end{aligned} \tag{5}$$

By examining equations (4) and (5) it is clear that since S_T and S_u are non negative constants, the return functions have the required monotonicity and equivalence properties. The sufficient conditions for optimality in a Markovian decision process have been met. Recursive application of equation (II-18) will yield $f_{ni}(x_n)$ that are optimum.

Replacing the various parts of equation (II-18) by their equivalent expressions from equations (1), (3), and (4) and considering equation (2) gives the following. If $n = 1$ and for all $1 \leq i \leq 3$, $x_1 \in S_{x_1}$,

$$f_{1i}(x_1) = \max_{0 \leq d_1 \leq x_1} \left[p_{i3}\ S_T\ r_H(d_1) \right] \tag{6}$$

Assuming that $r_H(d_n)$ is non decreasing in d_n, this becomes

$$f_{1i}(x_1) = p_{i3}\ S_T\ r_H(x_1) \tag{7}$$

If we replace the various factors of equation (II-18) by their equivalent expressions from equations (1), (3), and (5), and consider equation (2), the following is obtained. If $2 \leq n \leq N$ and for all $1 \leq i \leq 3$, $x_n \in S_{x_n}$,

$$f_{ni}(x_n) = \max_{0 \leq d_n \leq x_n} \sum_{j=1}^{2} p_{ij}\ S_T\ S_u\ f_{n-1,j}(x_n)$$

$$+ p_{i3} \left[S_T\ r_H(d_n) + S_T\ S_u\ f_{n-1,3}(x_n - d_n) \right] \tag{8}$$

which can be written as

$$f_{ni}(x_n) = S_T\, S_u \sum_{j=1}^{2} p_{ij}\, f_{n-1,j}(x_n)$$

$$+ S_T\, p_{i3} \max_{0 \leq d_n \leq x_n} \left[r_H(d_n) + S_u\, f_{n-1,3}(x_n - d_n) \right] \quad (9)$$

Equations (7) and (9) constitute the recursive relationships that will yield maximum expected hits and the corresponding optimum weapon allocation for the duel where target acquisition is considered. The optimum salvo size is a function of weapons available, the Markov state of the system, and the number of passes remaining. It will be denoted $d^*_{ni}(x_n)$.

A Simplifying Feature

When making calculations using equations (7) and (9), the following observation is helpful. These equations depend on i only because of p_{ij}. From the Markov state transition matrix, Figure 6, $p_{2j} = p_{3j}$ for all $1 \leq j \leq 3$. Thus, equations (7) and (9) have the following properties:

$$f_{n2}(x_n) = f_{n3}(x_n) \quad (10)$$

$$d^*_{n2}(x_n) = d^*_{n3}(x_n) \quad (11)$$

This fact allows considerable reduction in the amount of computation required to evaluate equations (7) and (9). It also eliminates one-third of the items of data that would otherwise have to be included when tabulating the results.

Note that while the foregoing observation is a convenience, it does not allow reduction in the number of Markov states. States 2 and 3 must both be included because transition to state 2 involves no weapon delivery while weapon delivery does accompany transition to state 3.

Concavity

The concavity property that was proved in Chapter III for equation (III-10) also holds for equation (9). In chapter III we showed that if $r(d_n)$ is concave in d_n and $f_{n-1}(x_{n-1})$ is concave in x_{n-1}, then the expression

$$\max_{0 \leq d_n \leq x_n} \left[r_H(d_n) + S_u \, f_{n-1}(x_n - d_n) \right]$$

is concave in x_n. Thus, if $r_H(d_n)$ and $f_{n-1,j}(x_{n-1})$ are concave in d_n and x_{n-1} respectively, since $S_T \geq 0$, $S_u \geq 0$, and $p_{ij} \geq 0$ for all i,j, $f_{ni}(x_n)$ as determined by equation (9) is concave in x_n for $1 \leq i \leq 3$, $2 \leq n \leq N$. Further, it is clear from equation (7) that if $r_H(x_1)$ is concave in x_1, then $f_{1i}(x_1)$ is concave in x_1 for all $x_1 \in S_{x_1}$, $1 \leq i \leq 3$. Thus, by mathematical induction, $f_{ni}(x_n)$ is concave in x_n for all $1 \leq n \leq N$, $1 \leq i \leq 3$, $x_n \in S_{x_n}$. This property can be used to reduce the amount of calculation that is required because it means that any local maximum that is found for equation (9) is also a global maximum.

Incorporating Multiple Modes of Attack

We will now introduce into the model the notion that the pilot need not make every pass in exactly the same way. In addition to

selecting the optimum salvo size, he must specify other quantities such as direction of approach, speed, dive angle, and pullout altitude. The aggregation of values taken by these other quantities on a given pass will be referred to as the "mode of attack."

Let the discrete variable, k_n, be the mode of attack index at pass n and let S_{k_n} indicate the set of all values that k_n can take. Thus at each pass, values for the two decision variables d_n and k_n must be chosen. These decision variables are the elements of the decision vector $D_n = (d_n, k_n)$.

For the most part, the modifications required to accommodate mode of attack are minor. A number of the quantities in the model depend on the value of k_n. The survival probabilities become $S_T(k_n)$ and $S_u(k_n)$. The Markov state transition probabilities become $p_{ij}(k_n)$. The salvo effectiveness function becomes $r_H(d_n, k_n) = r_H(D_n)$, the expected hits per salvo versus salvo size and mode of attack.

One nontrivial modification of the model is required. In the duel with probabilistic acquisition, we dealt at each stage with a functional equation of the form

$$f_{ni}(x_n) = \max_{d_n \in S_{d_n}(x_n)} \left\{ \sum_{j=1}^{3} p_{ij}\, g_{nij} \left[x_n, d_n, f_{n-1,j}(x_{n-1})\right] \right\} \quad (12)$$

The expression $\left\{ \cdot \right\}$ in equation (12) represents the expected hits for given n, i, x_n, and d_n. Since this expression represents the expected hits for a given value of n, and since S_T and S_u are constants, the probability of surviving the remaining n passes is at least

$(S_T\ S_u)^n$ for all $d_n \in S_{d_n}(x_n)$.[2] In effect, the $d^*_{ni}\ (x_n)$ that satisfy equation (12) maximize the expected hits subject to the constraint that the expected attrition on the remaining n passes does not exceed $1 - (S_T\ S_u)^n$, i.e., not more than n passes are to be made and the probability of surviving each pass is $(S_T\ S_u)$.

When multiple modes of attack are available at each pass, we must deal with a functional equation of the more general form

$$f_{ni}(X'_n) = \underset{D_n \in S_{D_n}(X'_n)}{\text{Max}} \left\{ \sum_{j=1}^{3} p_{ij}(k_n)\ g_{rij}\left[X'_n,\ D_n,\ f_{n-1,j}(X'_{n-1})\right]\right\} \quad (13)$$

Equation (13) is the same as the corresponding equation (II-18) except for having recognized that the Markov state transition probabilities depend only on the k_n component of D_n. Note that the residual state vector X'_n now appears in place of the variable x_n which appeared in equation (12).

The expression $\{\cdot\}$ in equation (13) represents the expected hits for given n, i, X'_n, and D_n. The approach at each stage will be to select $D^*_{ni}(X'_n)$ so as to maximize the expected hits subject to a constraint on the expected attrition. Let s_n denote the constraining probability of surviving the remaining n passes. A complete description of the state of the system at pass n now requires knowledge of the

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[2]Note that S_T and S_u could depend on n and these comments would still apply. The constraining value of the expected attrition for the remaining n passes would then be

$$1 - \prod_{n'=1}^{n} S_T(n')\ S_u(n') \quad , \text{ where } n' \text{ is a dummy variable.}$$

value of the Markov state variable, i, the number of weapons remaining, x_n, and the constraining probability of surviving the remaining n passes, s_n.[3] The residual state vector is

$$X'_n = (x_n, s_n) \tag{14}$$

Now consider the various parts of equation (13) as they apply to this model. The definitions of the Markov states are unchanged. The Markov state transition probabilities are given by Figure 6 except that all of the acquisition probabilities now depend on k_n.

The transition equations for the state variables are as follows. For all $1 \leq n \leq N$, $1 \leq i \leq 3$

$$\begin{aligned} x_{n-1} &= x_n \qquad ; \quad j = 1,2 \\ x_{n-1} &= x_n - d_n \qquad ; \quad j = 3 \end{aligned} \tag{15}$$

$$s_{n-1} = \frac{s_n}{S_T(k_n)\ S_u(k_n)} \qquad ; \quad \begin{matrix} 1 \leq j \leq 3 \\ k_n \in S_{k_n}(s_n) \end{matrix} \tag{16}$$

where $s_0 = 1.0$. Equations (15) are the same as equation (1). Equation (16) can be rationalized as follows.[4] Let $\hat{s}_n$ be the actual probability of surviving the remaining n passes. Thus, we require that

$$\hat{s}_n \geq s_n \qquad ; \quad 1 \leq n \leq N \tag{17}$$

---

[3]Note that in equation (12), s_n has a single value for given n, i.e., it takes the value $1 - (S_T\ S_u)^n$. Thus in equation (12), it is not necessary to explicitly state the value of s_n when defining the state of the system for given n.

[4]Equation (16) also follows from the expression of the probability of survival constraint, equation (A-7), that is given in Appendix A.

Now if $s_0 = 1.0$ and n' is a dummy variable,

$$\hat{s}_n = \prod_{n'=1}^{n} S_T(k_{n'})\ S_u(k_{n'}) = S_T(k_n)\ S_u(k_n)\ \hat{s}_{n-1} \quad ;\ 1 \leq n \leq N \tag{18}$$

So using equation (17),

$$s_n \leq \hat{s}_n = S_T(k_n)\ S_u(k_n)\ \hat{s}_{n-1} \tag{19}$$

or

$$\hat{s}_{n-1} \geq \frac{s_n}{S_T(k_n)\ S_u(k_n)} \equiv s_{n-1} \tag{20}$$

By inspection of equations (15) and (16) and considering the non negativity of d_n, we can define the following sets which have the property that $x_n \in S_{x_n}$, $d_n \in S_{d_n}(x_n)$, $s_n \in S_{s_n}$, and $k_n \in S_{k_n}(s_n)$ for all $1 \leq n \leq N$.

$$S_{x_n} = \left\{ x_n :\ x_n \in \left\{ 0,1,\text{---},\bar{x}_N \right\} \right\} \tag{21}$$

$$S_{s_n} = \left\{ s_n :\ \bar{s}_N \leq s_n \leq 1 \right\} \tag{22}$$

$$S_{d_n}(x_n) = \left\{ d_n :\ d_n \in \left\{ 0,1,\text{---},x_n \right\} \right\} \tag{23}$$

$$S_{k_n}(s_n) = \left\{ k_n :\ S_T(k_n)\ S_u(k_n) \geq s_n,\ k_n \in S_k \right\} \tag{24}$$

where $\bar{x}_N > 0$ and $0 \leq \bar{s}_N < 1.0$. The quantity $\bar{s}_N$ is the smallest value of the constraining survival probability that is of interest. It establishes the range of values of s_N over which solutions will be obtained.

The return functions are entirely analogous to those for the duel with probabilistic acquisition, equations (4) and (5). If $n = 1$ and for all $1 \leq i \leq 3$,

$$g_{1ij}\left[X_1', D_1, f_{o,j}(X_o')\right] = 0 \qquad ; \quad j = 1,2$$

$$= S_T(k_1)\, r_H(d_1,k_1) \qquad ; \quad j = 3 \tag{25}$$

If $2 \leq n \leq N$ and for all $1 \leq i \leq 3$,

$$g_{nij}\left[X_n', D_n, f_{n-1,j}(X_{n-1}')\right]$$

$$= S_T(k_n)\, S_u(k_n)\, f_{n-1,j}(x_{n-1},s_{n-1}) \qquad ; \quad j = 1,2$$

$$= S_T(k_n)\, r_H(d_n,k_n) + S_T(k_n)\, S_u(k_n)\, f_{n-1,j}(x_{n-1},s_{n-1}) \qquad ; \quad j = 3 \tag{26}$$

Now replacing the various parts of equation (II-18), or equation (13), by their equivalents from equations (15), (16), (23), (24), (25), (26), and considering equations (21) and (22) gives the functional equations for the duel with acquisition and multiple attack modes. If $n = 1$, $1 \leq i \leq 3$,

$$f_{1i}(x_1,s_1) = \max_{\substack{k_1 \in S_{k_1}(s_1) \\ d_1 \in S_{d_1}(x_1)}} \Sigma\, p_{i3}(k_1)\, S_T(k_1)\, r_H(d_1, k_1)$$

$$= \max_{k_1 \in S_{k_1}(s_1)} \left[p_{i3}(k_1)\, S_T(k_1)\, r_H(x_1, k_1)\right] \tag{27}$$

since $r_H(d_1, k_1)$ is assumed to be a monotonically non decreasing function of d_1.

If $2 \le n \le N$, $1 \le i \le 3$,

$$f_{ni}(x_n,s_n) = \underset{\substack{k_n \in S_{k_n}(s_n) \\ d_n \in S_{d_n}(x_n)}}{\text{Max}} \left\{ \sum_{j=1}^{2} p_{ij}(k_n) S_T(k_n) S_u(k_n) f_{n-1,j}\left(x_n, \frac{s_n}{S_T(k_n)S_u(k_n)}\right) \right. \tag{28}$$

$$\left. + \, p_{i3}(k_n)\left[S_T(k_n) r_H(d_n,k_n) + S_T(k_n)S_u(k_n) f_{n-1,3}\left(x_n - d_n, \frac{s_n}{S_T(k_n)S_u(k_n)}\right)\right]\right\}$$

Since the first term of $\left\{ \cdot \right\}$ in equation (28) is independent of d_n, we can write

$$f_{ni}(x_n,s_n) = \underset{k_n \in S_{k_n}(s_n)}{\text{Max}} \left\{ S_T(k_n) S_u(k_n) \sum_{j=1}^{2} p_{ij}(k_n) f_{n-1,j}\left(x_n, \frac{s_n}{S_T(k_n)S_u(k_n)}\right) \right. \tag{29}$$

$$\left. + \, S_T(k_n) p_{i3}(k_n) \underset{0 \le d_n \le x_n}{\text{Max}} \left[r_H(d_n,k_n) + S_u(k_n) f_{n-1,3}\left(x_n - d_n, \frac{s_n}{S_T(k_n)S_u(k_n)}\right)\right]\right\}$$

The optimal values of the decision variables will be denoted $d_{ni}^*(x_n,s_n)$ and $k_{ni}^*(x_n,s_n)$.

For the same reasons that were mentioned with respect to equations (7) and (9), equations (27) and (29) have the following properties:

$$f_{n2}(x_n,s_n) = f_{n3}(x_n,s_n) \tag{30}$$

$$d_{n2}^*(x_n,s_n) = d_{n3}^*(x_n,s_n) \tag{31}$$

$$k_{n2}^*(x_n,s_n) = k_{n3}^*(x_n,s_n) \tag{32}$$

Using a Discrete Attrition Constraint

Before considering an example of the application of equations (27) and (29), a practical difficulty must be faced. The quantity s_n has been treated as a continuous variable. Also, $S_T(k_n)$ and $S_u(k_n)$ can take any value from 0 to 1.0. Under these assumptions, the development is rigorous.

In making numerical calculations, however, s_n cannot be treated as a continuous variable when $f_{ni}(x_n, s_n)$ is tabulated. Accordingly, the interval $[\bar{s}_N, 1.0]$ will be divided into a discrete number, M, of increments of size Δs. At each stage, s_n is treated as a discrete variable taking only the values $\bar{s}_N + m\Delta s$ where $m = 0,1,2,\cdots,M$. This constitutes a modification of the definition of S_{s_n}, equation (22). The set S_{s_n} will now be defined as

$$S_{s_n} = \left\{ s_n : s_n \in \left\{ \bar{s}_N, \bar{s}_N + \Delta s, \bar{s}_N + 2\Delta s, \cdots, \bar{s}_N + M\Delta s \right\} \right\} ; \; 1 \leq n \leq N \tag{33}$$

We must also modify the transformation equation (16) because the quotient $s_n / S_T(k_n) S_u(k_n)$ will not in general produce a value of s_{n-1} such that $s_{n-1} \in S_{s_{n-1}}$ as defined by equation (33). Our approach is to select the next larger acceptable value for s_{n-1} or symbolically,

$$s_{n-1} = 1 - \left\langle \frac{1 - \dfrac{s_n}{S_T(k_n) S_u(k_n)}}{\Delta s} \right\rangle \Delta s \quad ; \; \begin{matrix} 1 \leq j \leq 3 \\ k_n \in S_{k_n}(s_n) \end{matrix} \tag{34}$$

where the symbol $\langle X \rangle$ means the largest integer value no greater than X. The definition of $S_{k_n}(s_n)$, equation (24), still applies.

To demonstrate that equation 34) will satisfy the survival constraint, write equation (34) as

$$\left\langle \frac{1 - \dfrac{s_n}{S_T(k_n)S_u(k_n)}}{\Delta s} \right\rangle = \frac{1 - s_{n-1}}{\Delta s} \tag{35}$$

Removing the largest integer value restriction, equation (35) becomes

$$\frac{1 - \dfrac{s_n}{S_T(k_n)S_u(k_n)}}{\Delta s} \geq \frac{1 - s_{n-1}}{\Delta s} \tag{36}$$

which becomes

$$s_n \leq S_T(k_n)S_u(k_n)\ s_{n-1} \tag{37}$$

or

$$s_{n-1} \geq \frac{s_n}{S_T(k_n)S_u(k_n)} \tag{38}$$

Thus, the s_{n-1} that is produced by equation (34) will always be at least as great as the s_{n-1} that is produced by equation (16).

It is well to note that by admitting only certain values of s_n, we have a more restrictive optimization than would result if s_n could be continuous. Thus, the maximum expected hits obtained from the discrete case cannot be greater than the maximum expected hits obtainable if s_n could be treated as continuous. The amount of discrepancy depends on the value of Δs. For a given problem, a

sensitivity analysis would be desirable to determine the extent to which the solution depends on the value of Δs.[5]

Numerical Example

To illustrate the application of equations (27) and (29), suppose an aircraft has eight bombs and can make a maximum of three passes at the target.

$N = 3$ passes

$x_N = 8$ bombs

Suppose further that acquisition is probabilistic on each pass with probabilities as shown in Table 3 and the corresponding matrix of Markov state transition probabilities as shown in Figure 7 for all k_n. This transition matrix can be computed from the values in Table 3 and by use of the Markov state transition probabilities in Figure 6.

---

[5]It is also possible to use an additive transformation relationship for the attrition constraint if s_n, $S_T(k_n)$, and $S_u(k_n)$ are close to 1.0. Suppose $A(k_n) \equiv 1.0 - S_T(k_n)\, S_u(k_n)$. Then equation (16) can be written

$$s_{n-1} = \frac{s_n}{1 - A(k_n)} = s_n \sum_{i=0}^{\infty} \left[A(k_n)\right]^i \approx s_n + s_n\, A(k_n)$$

if $A(k_n)$ is small. Further, if s_n is close to 1.0,

$$s_{n-1} \approx s_n + A(k_n)$$

which might be used in place of equation (34). Now if $S_T(k_n)$ and $S_u(k_n)$ are allowed only to take values such that $A(k_n) = n\Delta s$ where n is a positive integer and Δs is as used in equation (33), then use of the above additive survival constraint transformation avoids the truncation error that is introduced by equation (34). Where it is applicable, this procedure may be preferable to the use of equation (34) because the implications of restricting the values that the input parameters can take are perhaps easier to understand than the implications of the truncation that occurs in equation (34). This is especially true because the truncation error tends to be cumulative as successive transformations are performed.

TABLE 3

EXAMPLE ACQUISITION PROBABILITIES

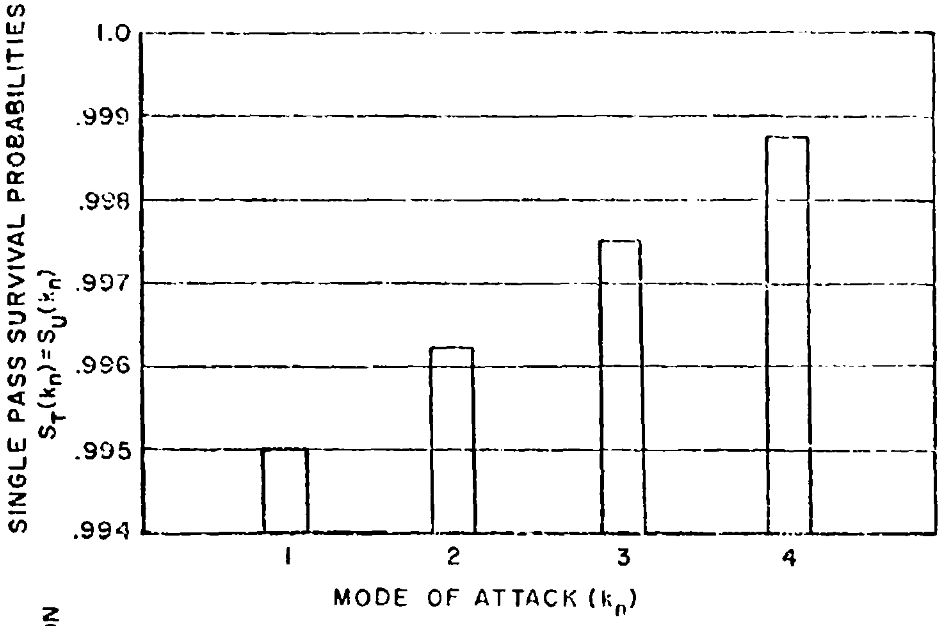
(All k_n)

$P(A_o)$	=	.725
$P(A_oD)$	=	.340
$P(A_1)$	=	.915
$P(A_1D)$	=	.775

	j = 1 A*	j = 2 AD*	j = 3 AD
i = 1: A*	0.275	0.385	0.340
i = 2: AD*	0	0.225	0.775
i = 3: AD	0	0.225	0.775

Fig. 7.--Example values of Markov state transition probabilities.

Let there be four modes of attack available on each pass where variation in mode of attack does not affect the acquisition probabilities but does have effect on both survival probabilities and salvo effectiveness. Suppose $S_T(k_n) = S_u(k_n)$ for all modes and for all $1 \leq n \leq N$. The values of the survival probabilities are given in Figure 8. It is also convenient to note the attrition per pass that is defined in the insert on Figure 8.



EXPECTED ATTRITION PER PASS $1-S_T(k_n)S_U(k_n)$	1	2	3	4
	0.0100	0.0075	0.0050	0.0025

Fig. 8.--Probability of survival and attrition per pass versus mode of attack.

Salvo effectiveness versus salvo size and mode of attack is given in Figure 9. These curves were generated by use of equation (III-18). The parameters Ψ and θ have the same meaning they did in Chapter III. Note that in Figure 9, Ψ is held constant and different salvo effectiveness functions are obtained by using different values of θ.

As the mode of attack index (k_n) increases from 1 to 4, the survival probabilities get progressively higher while the salvo effectiveness gets progressively lower. This type of situation might arise if increasing values of k_n correspond to increasing aircraft speed.

Calculations for this example were made with $\bar{x}_N = 8$ and $\bar{s}_N = 0.976$. Weapons are assumed to be allocated in groups of one. The survival constraint was varied in increments of $\Delta s = 0.002$.

The principal results are tables of $f_{ni}(x_n,s_n)$, $d_{ni}^*(x_n,s_n)$, and $k_{ni}^*(x_n,s_n)$. The tables that were generated for this example provide the foregoing information for any number of weapons remaining up to eight, for any attrition constraint up to .024 and for any number of passes remaining up to five. Tables 4, 5, and 6 are extracts taken from these tables at $n = 3$. The complete tables are contained in an unpublished computer printout that is currently in the possession of the author.

For a given n, each of these tables has three variables. The survival constraint is stated in terms of attrition $(1 - s_n)$. There are two entries in each block of each table. The first entry applies to Markov state 1 (target not yet acquired) and the second entry applies to Markov states 2 and 3 (target acquisition has occurred).

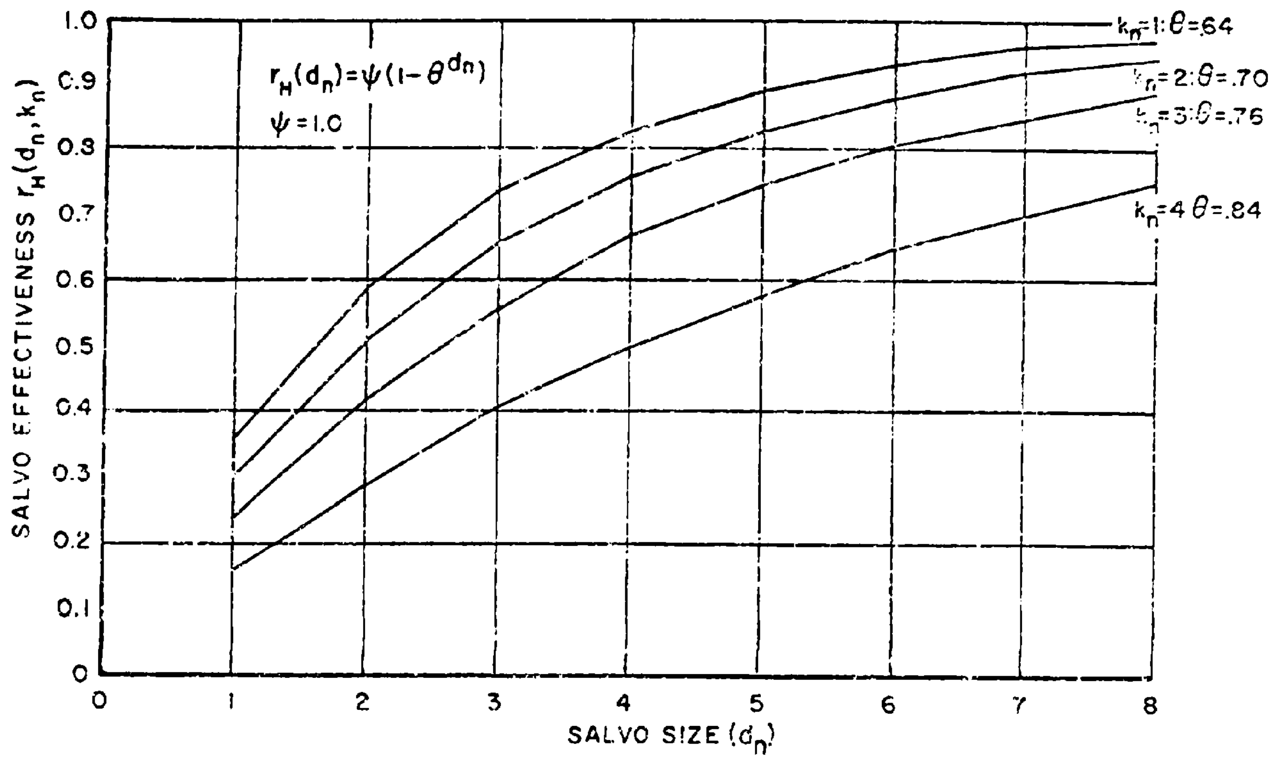


Fig. 9.--Salvo effectiveness versus salvo size and mode of attack.

TABLE 4

MAXIMUM EXPECTED HITS: $f_{3i}(x_3, s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining (x_3): 4	6	8
.006	.226	.274	.301
	.515	.624	.687
.012	.562	.709	.850
	.694	.908	1.066
.018	.724	.929	1.093
	.888	1.124	1.301
.024	.873	1.089	1.246
	1.014	1.278	1.509

TABLE 5

BEST SALVO SIZE: $d^*_{3i}(x_3, s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining (x_3): 4	6	8
.006	4	6	8
	4	6	8
.012	2	3	5
	2	3	4
.018	2	2	3
	2	3	3
.024	1	2	3
	2	3	3

TABLE 6

BEST MODE OF ATTACK: $k_{3i}^{*}(x_3, s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining (x_3): 4	6	8
.006	3 3	3 3	3 3
.012	4 3	4 3	4 3
.018	4 1	4 2	4 3
.024	4 1	4 1	3 2

Examination of Table 4 shows that the maximum expected hits increase with number of weapons available at a fixed attrition constraint and it increases as the attrition constraint is relaxed with a fixed number of weapons available. Also, the maximum expected hits is greater when target acquisition has occurred (Markov states 2 and 3) than when target acquisition has not yet occurred (Markov state 1). All these trends agree with intuition.

Table 5 indicates the best salvo size as a function of the state of the system. Note that these results indicate the number of weapons to deliver if target acquisition and weapon delivery occurs. If target acquisition and weapon delive does not occur, no weapons are to be delivered. From Table 5, the best salvo size increases as the number of available weapons increases with a fixed constraint on th.

expected attrition and decreases as the constraint on the expected attrition becomes less restrictive with a fixed number of weapons available. These trends agree with intuition.

By using the series of tables from which Tables 5 and 6 were extracted, a complete policy for the duel can be constructed. Such a policy is shown in Figure 10 for the example of Tables 4, 5, and 6.

Figure 10 shows three columns of blocks. The left hand column applies when $N = 3$ passes remain, the second column of blocks applies when $n = 2$ passes remain and the third column of blocks applies when $n = 1$ pass remains. Each block contains three entries on the left which define a state of the system. The entries on the right in each block indicate the best action corresponding to the state that is indicated by the entries on the left in the same block. The appearance of the symbol A indicates that target acquisition has occurred and the symbol A* indicates that target acquisition has not occurred. The arrows connecting the blocks show the possible transitions from state to state.

Figure 10 shows the policy for the case where the constraining value of expected attrition at the beginning of the duel is $1 - s_3 = 0.012$. When the first pass is made, acquisition has not yet occurred and eight weapons are available. The policy is to make the first pass in mode 4 and deliver five weapons if target acquisition and weapon delivery occurs. If target acquisition fails to occur on the first pass, then the system is in state A*, $x_2 = 8$, and $1 - s_2 = .008$ with 2 passes remaining; the best action is to use mode 4 with the intention of delivering five weapons. If target acquisition occurs but no weapons

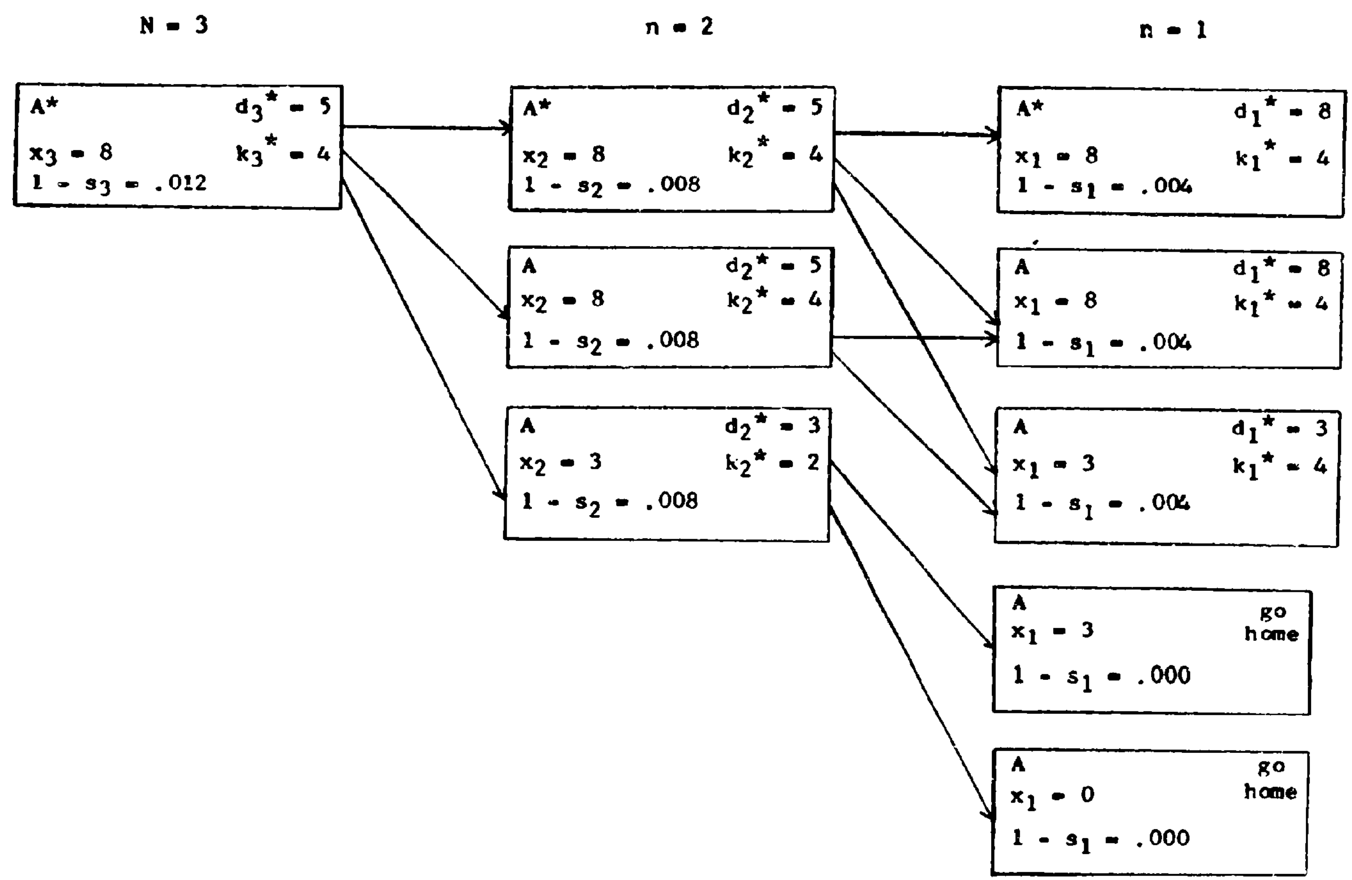


Fig. 10.--Selected attack policy for the E_H duel.

are delivered on the first pass, the system is in state A, $x_2 = 8$, and $1 - s_2 = .008$ with two passes remaining; the best action is to use mode 4 with the intention of delivering five weapons. If target acquisition and weapon delivery occurs on the first pass, then the system is in state A, $x_2 = 3$, and $1 - s_2 = .008$ with two passes remaining; the best action is to use mode 2 with the intention of delivering three weapons.

The following are some further comments about the results of the foregoing example. The decrease in best salvo size with less restrictive attrition constraint for fixed number of weapons available is an interesting phenomenon. We might think of the less restrictive attrition constraint as representing a less conservative attitude, i.e., more willing to take a chance. Now as our attitude becomes less conservative, we are less concerned about survival and more concerned about maximum effectiveness. The most conservative thing to do is to deliver all weapons on the first pass. As our attitude becomes less conservative, we depart further and further from this policy, i.e., the salvo sizes at a given pass become smaller thus leaving more weapons for future passes. As the attitude becomes less and less conservative, there comes a point where we, in effect, ignore survival altogether and simply maximize effectiveness. Relaxing the attrition constraint beyond this point would have no further effect on the policy relating to salvo size. Note that the $d_{3i}^*(x_3, s_3)$ values in Table 5 are beginning to reflect this phenomenon since the optimum salvo size is nearly the same at $1 - s_3 = .018$ and $1 - s_3 = .024$.

As a further comment, we note in Table 6 what seems to be a general outcome that is illustrated by this example. The best mode of attack is more conservative when acquisition has not yet occurred, i.e., the first entry in each block in Table 6 not less than the second entry in the block. This reflects the fact that the acquisition probabilities do not depend on mode of attack in this example while the survival probability and weapon effectiveness do depend on mode of attack. When acquisition has not yet occurred, the tendency seems to be to use a "safer" mode of attack and locate the target in order to save the aircraft for a less conservative attack on a subsequent pass when the target has already been acquired. Once acquisition has occurred, we tend to use the more effective mode of attack.

Having developed a generalized E_H duel and discussed some example results, we will close this chapter by noting that as with other duels the return-versus-attrition function is an important characterizing feature of the duel. Figure 11 shows this function for the foregoing example. The data for this figure were taken from the same table that the data for Table 4 were taken from. Figure 11 assumes that acquisition has not yet occurred (Markov state $i = 1$), eight weapons are available, and three passes can be made. A particular selected policy like that of Figure 10 applies to each of the points on the return-versus-attrition function.

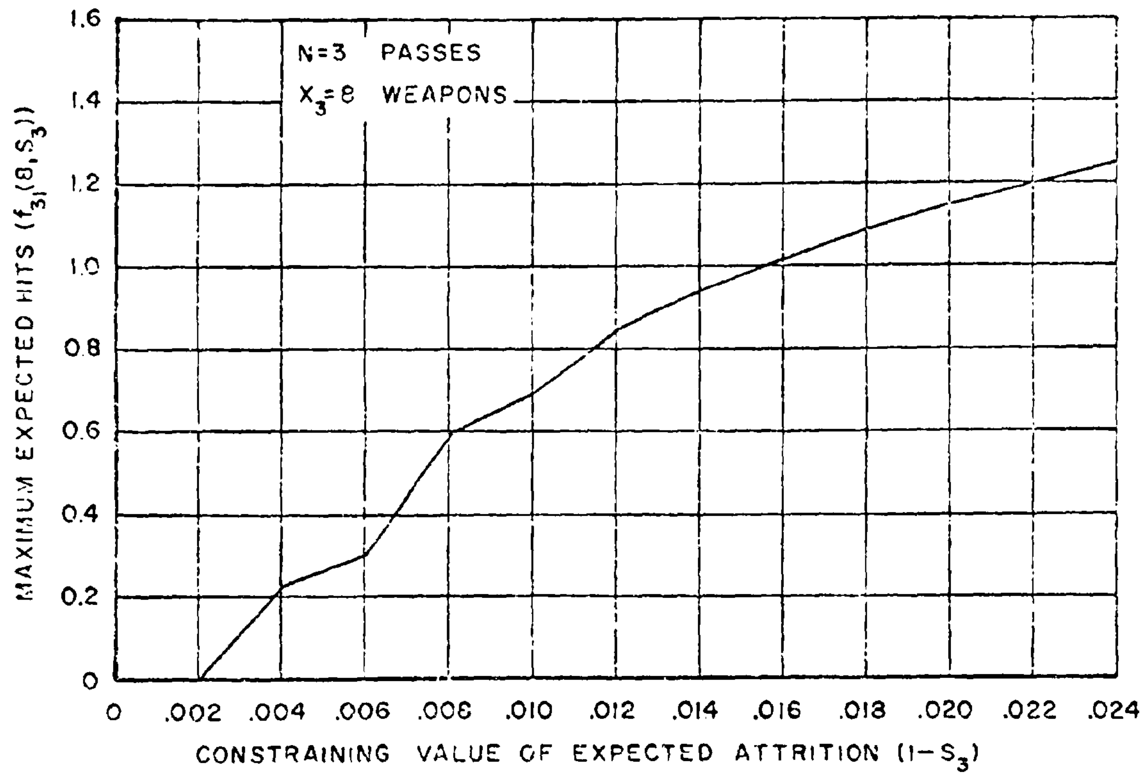


Fig. 11.--Return-versus-attrition function for the generalized E_H duel.

CHAPTER V

A DUEL WITH A DIFFERENT OBJECTIVE

The P_K Duel

Now consider an air-to-ground duel in which the objective is to hit the target at least once. This type of situation might arise in an attack on a small well fortified position requiring a direct hit. A near miss is assumed to do no damage and one hit is assumed to be adequate. In this discussion, the return will be referred to as "probability of kill" and the cost is in terms of the number of aircraft lost. This will be referred to as the "P_K duel." The model to be developed is quite similar to the model that was developed in Chapters III and IV.

Let us first study a simple duel analogous to that of Chapter III. In a simple P_K duel where probability of kill is the objective, the stage return, $r_K(d_n)$, is the probability of kill versus salvo size. The quantity $f_n(x_n)$ becomes the maximum probability of kill achievable with n passes and x_n weapons remaining.

The state variable transformation is given by equation (III-3). The sets S_{x_n} and $S_{d_n}(x_n)$ are defined by equations (III-4) and (III-5).

The return functions for this case are somewhat different from those of the model for dealing with expected hits. When one pass remains and d_1 weapons are to be delivered, the probability of kill is the probability of reaching the point of delivery times the probability

that the salvo scores at least one hit. Left over weapons have no value. Thus, if $n = 1$,

$$g_1[x_1, d_1, f_o(x_o)] = S_T\, r_K(d_1) \tag{1}$$

When n passes are yet to be made, the composition of the maximum n-1 stage return, $f_{n-1}(x_{n-1})$, with the stage n return, $r_K(d_n)$, can be accomplished as follows.

Suppose d_n weapons are to be delivered on pass n and the remaining $x_n - d_n$ weapons are to be delivered on the remaining n-1 passes. Target kill can occur only once and it can occur in one of two mutually exclusive ways. The target can be killed on pass n, or target kill can occur on one of the remaining n-1 passes. Since pass n chronologically precedes the remaining n-1 passes, if the target is killed on pass n, it cannot be killed on one of the n-1 remaining passes. The probability of target kill occurring on pass n is $S_T r_K(d_n)$. The probability that the target is not killed on pass n and that it is killed on one of the remaining n-1 passes is $S_T\, S_u[1-r_K(d_n)]\, f_{n-1}(x_{n-1})$. Since these two modes of target kill are mutually exclusive, we get the n stage probability of kill by adding the two terms. Thus for all $2 \leq n \leq N$,

$$g_n\,[x_n, d_n, f_{n-1}(x_{n-1})]$$
$$= S_T\, r_K(d_n) + S_T\, S_u\,[1 - r_K(d_n)]\, f_{n-1}(x_{n-1}) \tag{2}$$

By examining equation (1), we see that if $r_K(d_1)$ is defined for all $x_1 \in S_{x_1}$ and $d_1 \in S_{d_1}(x_1)$, then the return function for n=1 is defined for all $x_1 \in S_{x_1}$ and $d_1 \in S_{d_1}(x_1)$. Furthermore, when $2 \leq n \leq N$, we see from equation (2) that if $r_K(d_n)$ is defined for all $x_n \in S_{x_n}$

and $d_n \in S_{d_n}(x_n)$, then the return function meets the monotonicity requirement that is defined in Chapter II for a deterministic decision process. The previous statement is true because S_T, $S_u \geq 0$ and since $r_K(d_n)$ is a probability, then $1 - r_K(d_n) \geq 0$. Thus, application of the basic functional equation for a deterministic decision process, equation (II-5), will yield optimum values of $f_n(x_n)$ for all $1 \leq n \leq N$.

Substituting equations (1) and (2) above along with the appropriate transformations and set definitions, equations (III-3), (III-4) and (III-5) into equation (II-5) we get the following recursive relations for the simple duel with probability of kill as an objective.

If $n = 1$, $x_1 \in S_{x_1}$,

$$f_1(x_1) = \max_{0 \leq d_1 \leq x_1} \left[S_T \; r_K(d_1)\right] = S_T \; r_K(x_1) \tag{3}$$

since $r_K(d_n)$ is assumed to be monotonically nondecreasing.

If $2 \leq n \leq N$, $x_n \in S_{x_n}$,

$$f_n(x_n) = S_T \max_{0 \leq d_n \leq x_n} \left\{r_K(d_n) + S_u\left[1 - r_K(d_n)\right] f_{n-1}(x_n - d_n)\right\} \tag{4}$$

Nonrecursive Form

Let $R_n(d_n, d_{n-1}, \cdots, d_1)$ denote the probability of kill obtainable on passes $n, \cdots, 1$ for the allocation $(d_n, d_{n-1}, \cdots, d_1)$. With one pass remaining, we have

$$R_1(d_1) = S_T \; r_K(d_1) \tag{5}$$

With two passes remaining, we have

$$R_2(d_2,d_1) = S_T\ r_K(d_2) + S_T\ S_u \left[1 - r_K(d_2)\right] R_1(d_1)$$

$$= S_T\ r_K(d_2) + S_T^2\ S_u \left[1 - r_K(d_2)\right] r_K(d_1) \qquad (6)$$

With three passes remaining, we have

$$R_3(d_3,d_2,d_1) = S_T\ r_K(d_3) + S_T\ S_u \left[1 - r_K(d_3)\right] R_2(d_2,d_1)$$

$$= S_T\ r_K(d_3) + S_T^2\ S_u \left[1 - r_K(d_3)\right] r_K(d_2)$$

$$+ S_T^3\ S_u^2 \left[1 - r_K(d_3)\right]\left[1 - r_K(d_2)\right] r_K(d_1) \qquad (7)$$

We can now construct the expression for the N stage return for a given allocation, $(d_N,d_{N-1},---,d_1)$.

$$R_N(d_N,d_{N-1},---,d_1) = S_T \sum_{i=1}^{N} (S_T\ S_u)^{i-1} \prod_{n=N-i+2}^{N} \left[1-r_K(d_n)\right] r_K(d_{N-i+1}) \qquad (8)$$

where for an arbitrary function of n, say g(n),

$$\prod_{n=N+1}^{N} g(n) \equiv 1.0 \qquad (9)$$

Considering the constraints, equations (III-1) and (III-2), we can state the optimization problem as follows.

$$f_N(x_N) = \underset{d_N,---,d_1}{\mathrm{Max}} \left\{ S_T \sum_{i=1}^{N} (S_T S_u)^{i-1} \prod_{n=N-i+2}^{N} \left[1-r_K(d_n)\right] r_K(d_{N-i+1}) \right. \qquad (10)$$

subject to

$$\sum_{n=1}^{N} d_n \leq x_N \tag{11}$$

$$d_n \geq 0 \quad ; \quad 1 \leq n \leq N \tag{12}$$

Wr :ing the problem in the form of equations (10), (11), and (12) is perhaps helpful in understanding the origin of the recursive equations. It also promotes an appreciation of the simplicity of the recursive solution method. The development of nonrecursive problem statements and the derivation of recursive relationships therefrom are shown in Appendix A for some of the more general problems that are treated herein.

The Special Case where $\Psi = 1.0$

Equations (10), (11), and (12) are not very encouragin- from the standpoint of classical optimization techniques, however, we can gain one useful bit of insight by studying the recursive relations, equations (3) and (4). We will show that if $\Psi = 1.0$ and other conditions are appropriate, all weapons should be delivered on the first pass.

Suppose the stage return can be expressed as

$$r_K(d_n) = 1 - \theta^{dn} \quad ; \quad 0 \leq \theta \leq 1 \tag{13}$$

This is the same as equation (III-18) with $\Psi = 1.0$. Then from equation (3), when $n = 1$,

$$f_1(x_1) = S_T \ (1 - \theta^{x_1}) \tag{14}$$

We will now hypothesize that

$$f_{n-1}(x_{n-1}) = S_T\ (1 - \theta^{x_{n-1}}) = S_T\ (1 - \theta^{x_n - d_n}) \tag{15}$$

Then from equation (4)

$$f_n(x_n) = S_T \max_{0 \leq d_n \leq x_n} \left\{ 1-\theta^{d_n} + S_u\ \theta^{d_n}\ S_T(1-\theta^{x_n - d_n}) \right\} \tag{16}$$

or

$$f_n(x_n) = S_T \max_{0 \leq d_n \leq x_n} \left\{ 1-\theta^{d_n}\ (1-S_T S_u) - S_T S_u\ \theta^{x_n} \right\} \tag{17}$$

Let $\left\{ \cdot \right\} = Q(x_n, d_n)$. Then we note that

$$Q(x_n, d_n + 1) - Q(x_n, d_n) = -\ (1 - S_T S_u)(\theta - 1)^{d_n} \tag{18}$$

Since $S_T S_u \leq 1.0$ and $\theta \leq 1.0$, the above difference is non negative. It follows that $d_n^* = x_n$ and therefore substituting in equation (17) gives

$$f_n(x_n) = S_T\ Q(x_n, x_n) = S_T\ (1-\theta^{x_n}) \tag{19}$$

Thus, when $r_K(d_n)$ is given by equation (13), we have shown that $f_1(x_1)$ has the form of equation (19) and if $f_{n-1}(x_{n-1})$ has the form of equation (19), then so does $f_n(x_n)$; so by mathematical induction, for all $1 \leq n \leq N$, $d_n^* = x_n$ and $f_n(x_n)$ has the form of equation (19).

The conclusion is that in the P_K duel, if $r(d_n)$ has the form of equation (13), then all weapons should be delivered on the first pass, i.e., take $d_N^*(x_N) = x_N$.

Parametric Investigation

The parametric investigation is not as convenient for the P_K duel as it was for the E_H duel of Chapter III. If we assume $r_K(d_n)$ can be expressed by the analytic form, equation (III-18), the P_K duel objective function, equation (10) becomes

$$f_N(x_N) = \Psi\, S_T \max_{d,---,d} \left\{ \sum_{i=1}^{N} (S_T S_u)^{i-1} \prod_{n=N-i+2}^{N} \left[1-\Psi(1-\theta^{d_n})\right](1-\theta^{d_{N-i+1}}) \right\} \tag{20}$$

From equation (20), the optimal allocation depends on the three quantities Ψ, θ, and $(S_T S_u)$ for given values of x_N and N. In the parametric investigation of the E_H duel, the optimal allocation depended only on θ and $(S_T S_u)$ for given x_N and N and was independent of the value of Ψ.

Figure 3 illustrates the $r_K(d_n)$ function for $\Psi = 1.0$ and for the various values of θ that are used here. Since Ψ is a multiplicative constant, the curves of Figure 3 can be made to apply for any Ψ by simply changing the ordinate scale.

Tables 7 through 10 show the optimal allocation versus $(S_T S_u)$ and θ for various values of Ψ. Table 10 reflects the result for the special case where $\Psi = 1.0$. The entry in each block in these tables is the vector $(d_N^*,---,d_1^*)$ where zeros are omitted. It is clear in the P_K duel as it was in the E_H duel that if $n_1 \geq n_2$, then $d_{n_1}^* \geq d_{n_2}^*$ because of the discounting structure and the fact that the functional form of $r_K(d_n)$ is the same for all n.

TABLE 7

TABLE 8

OPTIMUM WEAPON ALLOCATION, $(d_N^*,---,d_1^*)$ vs. Ψ, θ, (S_uS_T)

$N = 8$ passes $x_N = 8$ weapons

$\Psi = 0.25$

Probability of Surviving one Pass (S_TS_u) \ θ	.76	.84	.92	.98
1.0				
.98	1,1,1,1, 1,1,1,1	2,1,1,1, 1,1,1	2,2,1,1, 1,1	4,3,1
.92	2,2,2,1, 1	3,2,2,1	4,3,1	7,1
.86	3,2,2,1	4,3,1	5,3	8
.80	3,2,2,1	4,3,1	6,2	8

$\Psi = 0.50$

Probability of Surviving one Pass (S_TS_u) \ θ	.76	.84	.92	.98
1.0				
.98	2,1,1,1, 1,1,1	2,2,1,1, 1,1	3,2,1,1, 1	4,3,1
.92	3,2,2,1	4,2,1,1	4,3,1	8
.86	4,3,1	4,3,1	6,2	8
.80	4,3,1	5,3	6,2	8

Each block contains the non-zero elements of $(d_N^*,---,d_1^*)$.

TABLE 9 TABLE 10

OPTIMUM WEAPON ALLOCATION, $(d_N^*,\text{---},d_1^*)$ vs. Ψ , θ, $(S_T S_u)$

N = 8 passes x_N = 8 weapons

$\Psi = 0.75$

Probability of Surviving one Pass $(S_T S_u)$	θ = .76	.84	.92	.98
1.0				
.98	2,2,1,1,1,1	3,2,1,1,1	4,2,1,1,	6,2
.92	4,3,1	4,3,1	6,2	8
.86	5,2,[illegible]	[illegible]3	7,1	8
.80	5,3	6,2	8	8

$\Psi = 1.0$

Probability of Surviving one Pass $(S_T S_u)$	θ = .76	.84	.92	.98
1.0				
.98	8	8	8	8
.92	8	8	8	8
.86	8	8	8	8
.80	8	[illegible]	8	8

Each block contains the non-zero elements of $(d_N^*,\text{---},d_1^*)$.

We see that with other factors held constant, increasing survival probability calls for more uniform distribution of weapons among the passes. The same type of trend occurs when θ or Ψ decreases in value with other factors held constant.

The difference between the E_H duel results and the P_K duel results can be appreciated by comparing Table 10 with Table 2. Ignore for the moment the different interpretation of $r(d_n)$ in the two models. The numerical inputs are the same for both of these sets of results. With the same numerical input values, the E_H duel and the P_K duel can have quite different optimal allocations.

Generalizing the P_K Duel

A model for the P_K duel with probabilistic acquisition and multiple modes of attack will now be developed. This model is similar to the model for the analogous E_H duel that was discussed in Chapter IV.

The salvo effectiveness function will be $r_K(D_n) = r_K(d_n, k_n)$; it will represent the probability of at least one hit as a function of salvo size and mode of attack. The maximum n stage return will be $f_{ni}(X_n) = f_{ni}(x_n, s_n)$; it will represent the maximum probability of at least one hit when the system is in Markov state i, n passes remain, x_n weapons remain, and the probability of surviving the remaining n passes must be at least s_n.

The following aspects of this model are identical to the corresponding aspects of the model for the E_H duel with probabilistic acquisition and multiple modes of attack that was developed in Chapter IV. The Markov state definitions are the same as those given in Chapter IV. The Markov state transition probabilities are given in

Figure 6. The basic transformation equations for the residual state variables are given by equations (IV-15) and (IV-16). The basic sets S_{x_n}, S_{s_n}, $S_{d_n}(x_n)$ and $S_{k_n}(s_n)$ are defined by equations (IV-21) through (IV-24). The required modification of the transformation equations and the set definitions to account for the discrete nature of numerical calculation related to s_n are given by equations (IV-34) and (IV-33), respectively.

The return functions for this model have the same form as those for the simple P_K duel but they must be modified to account for the presence of multiple acquisition states and multiple modes of attack.

When one pass remains, weapon delivery is associated with transition to Markov state 3 and weapons are not delivered when transition is to Markov states 1, or 2. The probability of surviving to the point of weapon release and the salvo effectiveness depend on the mode of attack. Thus, if $n = 1$ and $1 \leq i \leq 3$,

$$g_{1ij}\left[X_1', D_1, f_{o,j}(X_o')\right] \tag{21}$$

$$\begin{aligned} &= 0 \qquad ; \qquad j = 1,2 \\ &= S_T(k_1)\ r_K(d_1,k_1) \qquad ; \qquad j = 3 \end{aligned}$$

When more than one pass remains, the foregoing statements still apply. Furthermore, we can use the same argument that we used in relation to the simple P_K duel to justify an expression for the composition of the stage n return with the maximum return for the remaining n-1 stages.

If transition is to Markov state 3, weapon delivery is implied and for given X_n' and D_n, the probability of target kill occurring on

pass n is $S_T(k_n)\ r(d_n,k_n)$. The probability that the aircraft survives pass n, fails to kill the target on pass n, and then kills the target on one of the n-1 remaining passes is given by

$$S_T(k_n)\ S_u(k_n)\ [1 - r_K(d_n,k_n)]\ f_{n-1,3}(x_{n-1},s_{n-1})$$

If transition is to Markov states 1 or 2, no weapons are delivered on pass n and the probability that target kill occurs on pass n is zero. The probability that the aircraft survives pass n and kills the target on one of the remaining n-1 passes is $S_T(k_n)\ S_u(k_n)\ f_{n-1,j}(x_{n-1},s_{n-1})$ where x_{n-1} and s_{n-1} are given by the appropriate transformation equations. We can summarize the foregoing as follows. If $2 \le n \le N$ and for all $1 \le i \le 3$,

$$\begin{aligned} g_{nij}\,[X_n',D_n,f_{n-1,j}(X_{n-1})] &= S_T(k_n)S_u(k_n)f_{n-1,j}(x_{n-1},s_{n-1}) \ ;\ j = 1,2 \\ &= S_T(k_n)r_K(d_n,k_n) + S_T(k_n)S_u(k_n)\,[1-r_K(d_n,k_n)]\,f_{n-1,3}(x_{n-1},s_{n-1}) \ ;\ j=3 \end{aligned} \tag{22}$$

If $r_K(d_n,k_n)$ is defined for all $x_n \in S_{x_n}$, $d_n \in S_{d_n}(x_n)$, $1 \le n \le N$, then these return functions satisfy the sufficient conditions for optimality as defined in Chapter II. This is trivially true for equation (21). Equation (22) has the monotonicity property for all n, i, j, X_n', and D_n' in their respective sets because under all conditions

$$S_T(k_n)\ S_u(k_n)\ [1 - r_K(d_n,k_n)] \ge 0 \tag{23}$$

Likewise, the equivalence condition is satisfied because $g_{nij}\,[\cdot]$ is a linear function of $f_{n-1,j}(x_{n-1},s_{n-1})$.

The various parts are now available so the P_K duel with probabilistic acquisition and multiple modes of attack can be optimized by using the functional equation for a Markovian decision process, equation (II-18). Substituting values gives the following. In stating these equations, we will ignore the practical problem that s_n cannot be treated as a continuous variable when making numerical calculations. It will be understood that the appropriate modifications are used when calculations are made.

If $n = 1$ and for all $1 \leq i \leq 3$,

$$f_{1i}(x_1,s_1) = \max_{\substack{0 \leq d_1 \leq x_1 \\ k_1 \in S_{k_1}(s_1)}} \sum_{j=1}^{3} p_{ij}(k_1)\, S_T(k_1)\, r_K(d_1,k_1)$$

$$= \max_{k_1 \in S_{k_1}(s_1)} \left[p_{13}(k_1)\, S_T(k_1)\, r_K(x_1,k_1)\right] \qquad (24)$$

if we assume that $r_K(d_1,k_1)$ is a monotonically non decreasing function of d_1.

If $2 \leq n \leq N$ and for all $1 \leq i \leq 3$,

$$f_{ni}(x_n,s_n) = \max_{\substack{0 \leq d_n \leq x_n \\ k_n \in S_{k_n}(x_n)}} \left\{ \sum_{j=1}^{2} p_{ij}(k_n) S_T(k_n) S_u(k_n) f_{n-1,j}\left(x_n, \frac{s_n}{S_T(k_n)S_u(k_n)}\right) \right.$$

$$+ p_{13}(k_n) \Big[S_T(k_n) r_K(d_n,k_n) + S_T(k_n) S_u(k_n) \big[1 - r_K(d_n,$$

$$\left. f_{n-1,3}\left(x_n - d_n, \frac{s_n}{S_T(k_n)S_u(k_n)}\right)\big]\Big] \right\} \qquad (25)$$

Equation (25) can be simplified to

$$f_{ni}(x_n,s_n)=\max_{k_n\in S_{k_n}(s_n)}\left\{S_T(k_n)S_u(k_n)\sum_{j=1}^{2}p_{ij}(k_n)f_{n-1,j}\left(x_n,\frac{s_n}{S_T(k_n)S_u(k_n)}\right)\right.$$

$$+\ S_T(k_n)p_{i3}(k_n)\max_{0\le d_n\le x_n}\Big[r_K(d_n,k_n)+S_u(k_n)\left[1-r_K(d_n,k_n)\right]$$

$$\left.f_{n-1,3}\left(x_n-d_n,\frac{s_n}{S_T(k_n)S_u(k_n)}\right)\Big]\right\} \tag{26}$$

Numerical Example

The application of equations (24) and (26) will be illustrated by using a problem that is very nearly the same as the example problem of Chapter IV. In this example, $N = 3$ and $x_N = 8$. Acquisition is probabilistic and is characterized by the values given in Table 3 and Figure 7. The survival probabilities are given in Figure 8.

The salvo effectiveness function, $r_K(D_n)$, used for this problem differs from the one used in the example of Chapter IV. First, its interpretation is different since $r_K(D_n)$ is the probability of target kill versus salvo size and mode of attack. Second, the functional form of $r_K(D_n)$ is the same as before, i.e., equation (III-18) and the values of θ for the four modes of attack are unchanged, but we now have $\Psi = 0.25$. The values of Ψ and θ and the resulting $r_K(d_n,k_n)$ functions for the four modes of attack are shown in Figure 12.

Calculations were made with $\bar{x}_N = 8$ and $\bar{s}_N = 0.976$. Weapons are assumed to be allocated in groups of one. The survival constraint was

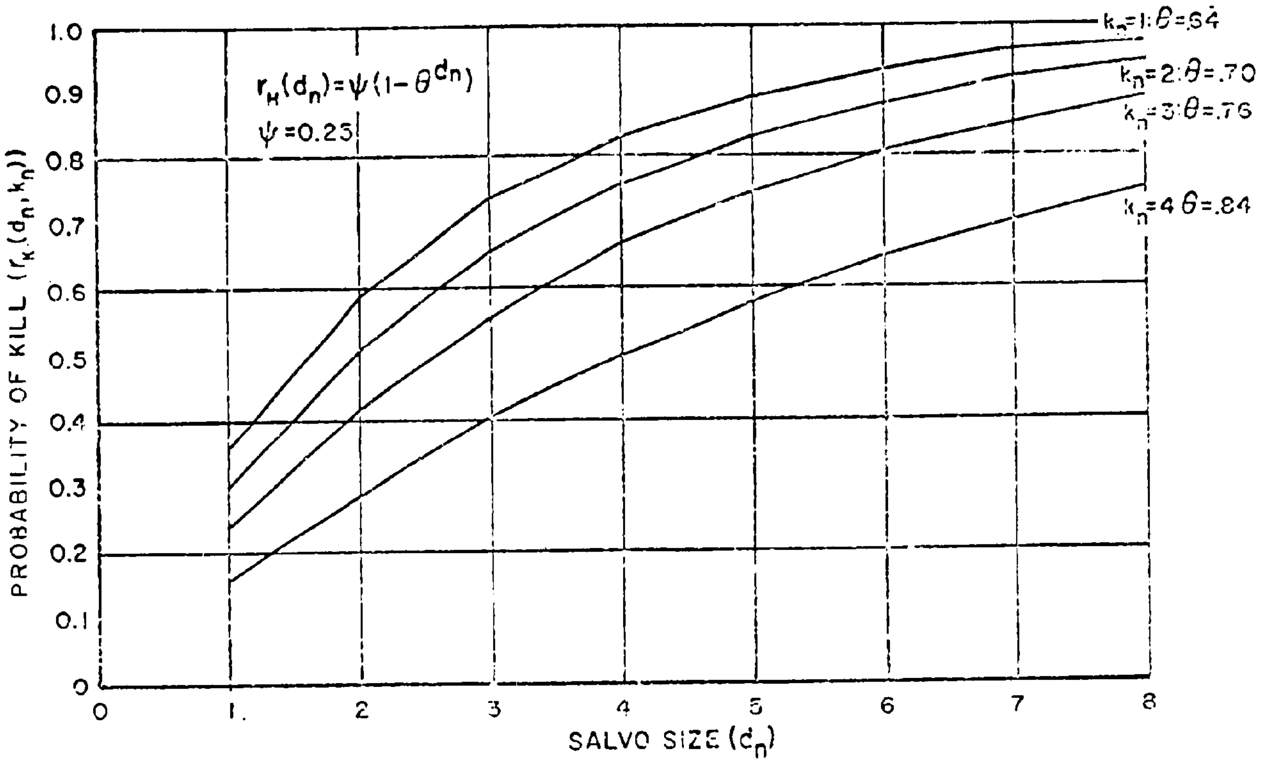


Fig. 12.--Salvo effectiveness versus salvo size and mode of attack.

varied in increments of $\Delta s = 0.002$. These values are the same as they were in the example of Chapter IV.

Tables 11, 12, and 13 show extracts from the principal tables of results. These tables respectively show maximum probability of kill, best salvo size, and best mode of attack versus number of weapons remaining, attrition constraint, and Markov state. Note that here as before, the first entry in each block in Tables 11, 12, and 13 applies when the system is in Markov state 1 (target acquisition has not yet occurred). The second entry in each block applies when the system is in Markov state 2 or 3 (target acquisition has occurred).

Figure 13 shows an illustrative optimum attack policy corresponding to $1 - s_3 = 0.012$. This figure was constructed from data in the series of tables of results that Tables 11, 12, and 13 were extracted from. The optimum attack policy of Figure 13 happens to be identical to the optimum attack policy of Figure 10 which applies to the example of Chapter IV. Note, however, that not all the results are the same for the two examples as can be seen by comparing Tables 11, 12, and 13 of this chapter with Tables 4, 5, and 6 of Chapter IV.

The return-versus-attrition function for this example can be read from the tables of results at $n = 3$, $x_3 = 8$, and assuming the system is in Markov state 1 (the target has not yet been acquired). Some of these values appear in the appropriate positions in Table 11. Figure 14 shows the resulting return-versus-attrition function for this example problem.

TABLE 11

MAXIMUM PROBABILITY OF KILL: $f_{3i}(x_3, s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining: (x_3) 4	6	8
.006	.057 .129	.068 .156	.075 .172
.012	.138 .168	.172 .216	.202 .251
.018	.176 .211	.219 .261	.253 .298
.024	.207 .238	.252 .294	.284 .334

TABLE 12

BEST SALVO SIZE: $d^*_{3i}(x_3, s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining: (x_3) 4	6	8
.006	4 4	6 6	8 8
.012	2 3	3 4	5 5
.018	1 3	3 3	3 5
.024	0 2	1 3	3 3

TABLE 13

BEST MODE OF ATTACK: $k_{3i}^{*}(x_3, s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining: (x_3) 4	6	8
.006	3 3	3 3	3 3
.012	4 2	4 3	4 3
.018	4 1	4 2	4 2
.024	4 1	4 1	3 2

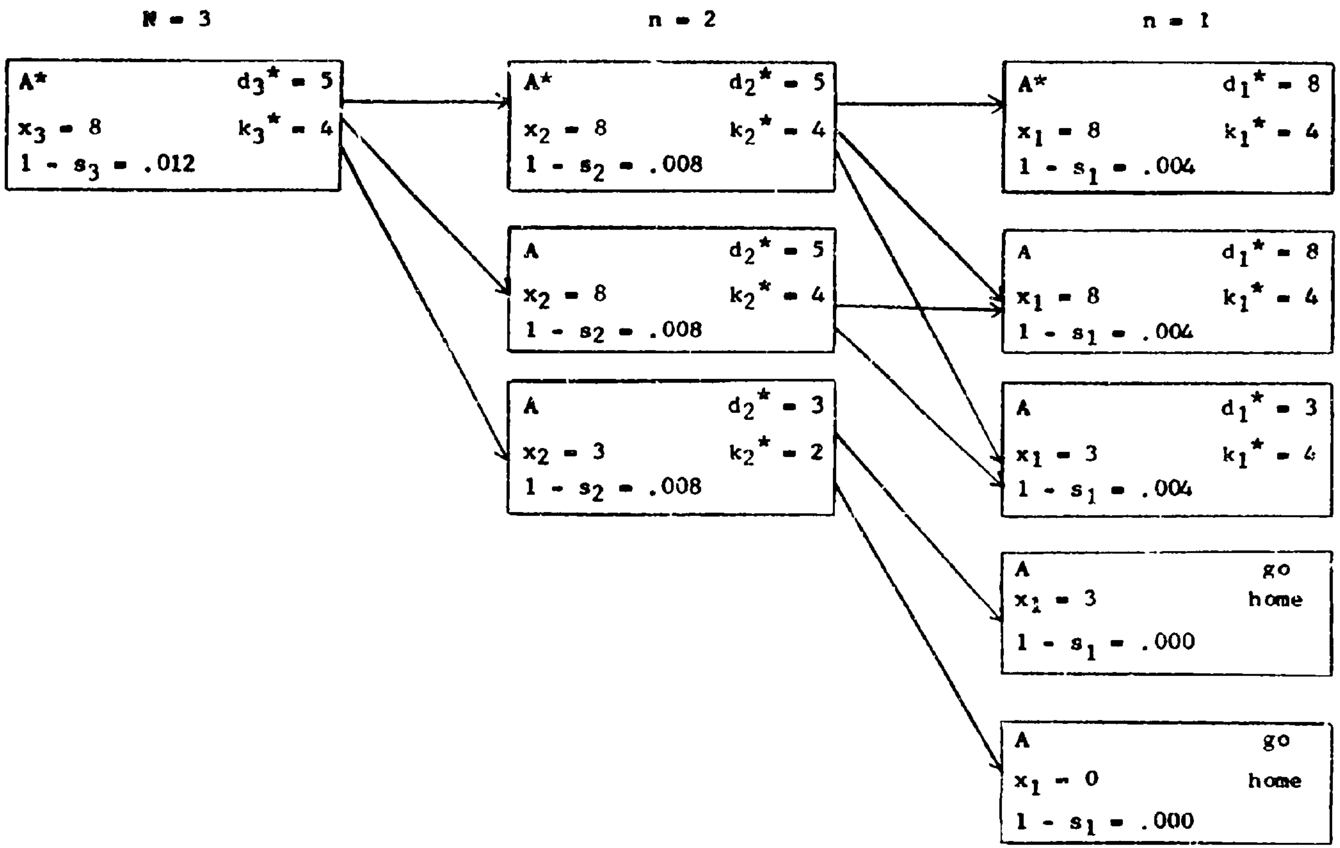


Fig. 13.--Selected attack policy for the P_K duel (1 - s = 0.012).

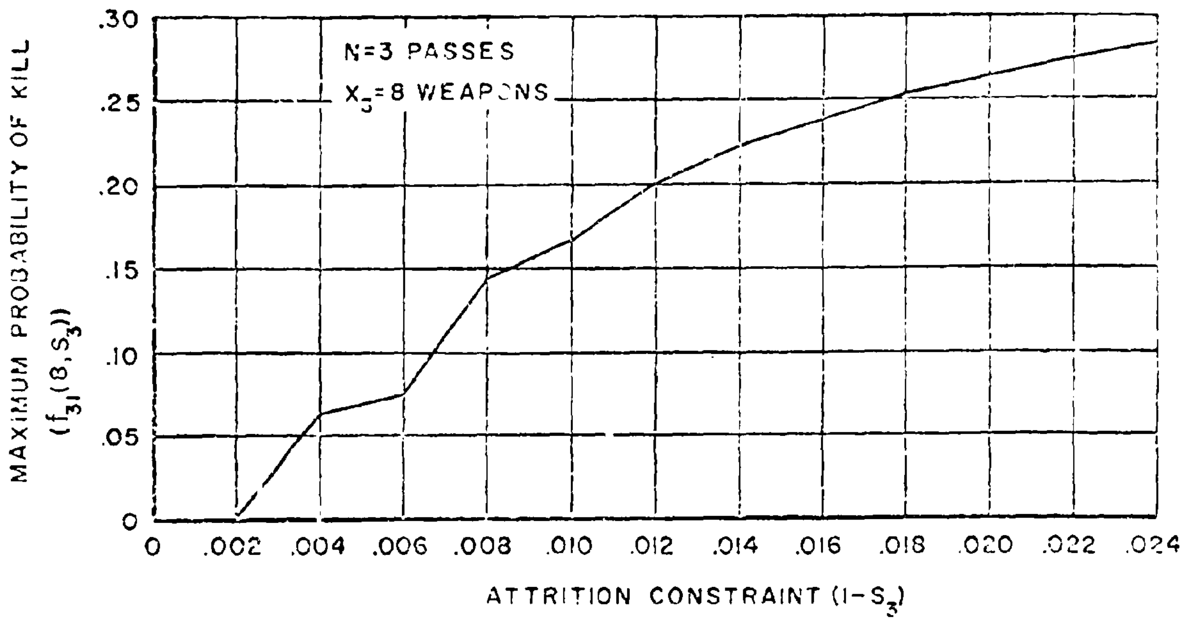


Fig. 14.--Return-versus-attrition function for the generalized P_K duel.

Numerical Example: Reduced Salvo Effectiveness

For another example of the application of the P_K duel model, equations (24) and (26), an interesting outcome results if the salvo effectiveness is made small while the acquisition and survival inputs are maintained the same as they were in the previous example, i.e., as in Figures 7 and 8. If the salvo effectiveness functions are as shown in Figure 15, the optimum policy at $N = 3$ is illustrated by the extracts shown in Tables 14 and 15.

The interesting feature is that for all values of attrition constraint greater than or equal to 0.012, zero weapons are delivered on the first pass. Note also that the "safest" mode of attack is employed, i.e., $k_3^* = 4$.

The complete attack policy for $1 - s_3 = 0.018$ is diagrammed in Figure 16. This policy says to make the first pass using the "safest" mode with no intention of delivering weapons. The purpose of the first pass is to acquire the target. Once target acquisition has occurred, the least conservative mode of attack, i.e., the most effective, is employed and all weapons are delivered. If the target is not acquired on the first pass, the process is repeated on the second pass.

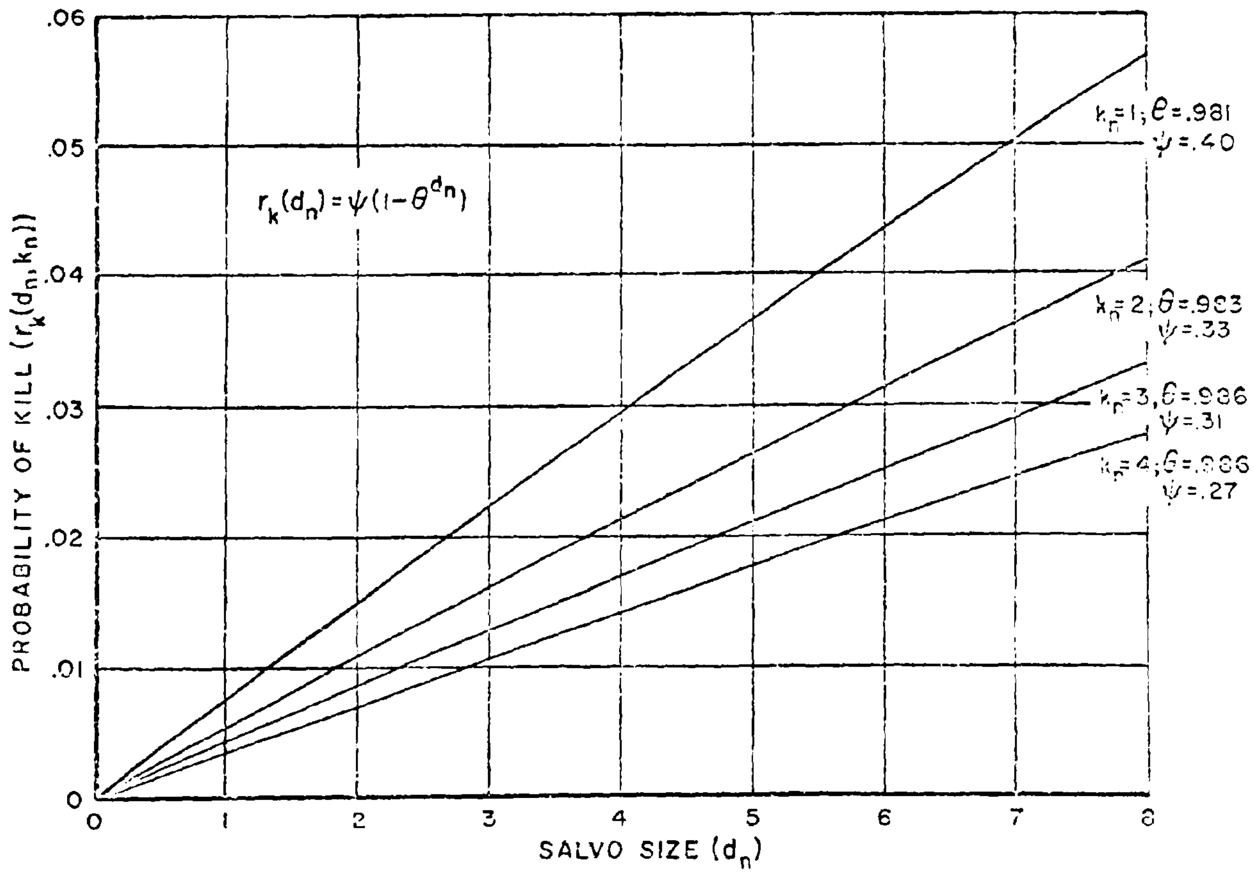


Fig. 15.--Salvo effectiveness functions: Reduced salvo effectiveness.

TABLE 14

BEST SALVO SIZE WITH REDUCED SALVO EFFECTIVENESS: $d_{3i}(x_3,s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining: (x_3) 4	6	8
.006	4 4	6 6	8 8
.012	0 4	0 6	0 8
.018	0 4	0 6	0 8
.024	0 4	0 6	0 8

TABLE 15

BEST MODE OF ATTACK WITH REDUCED SALVO EFFECTIVENESS:
$k_{3i}(x_3,s_3)$

Attrition Constraint $(1 - s_3)$	Number of Weapons Remaining: (x_3) 4	6	8
.006	3 3	3 3	3 3
.012	4 1	4 1	4 1
.018	4 1	4 1	4 1
.024	4 1	4 1	4 1

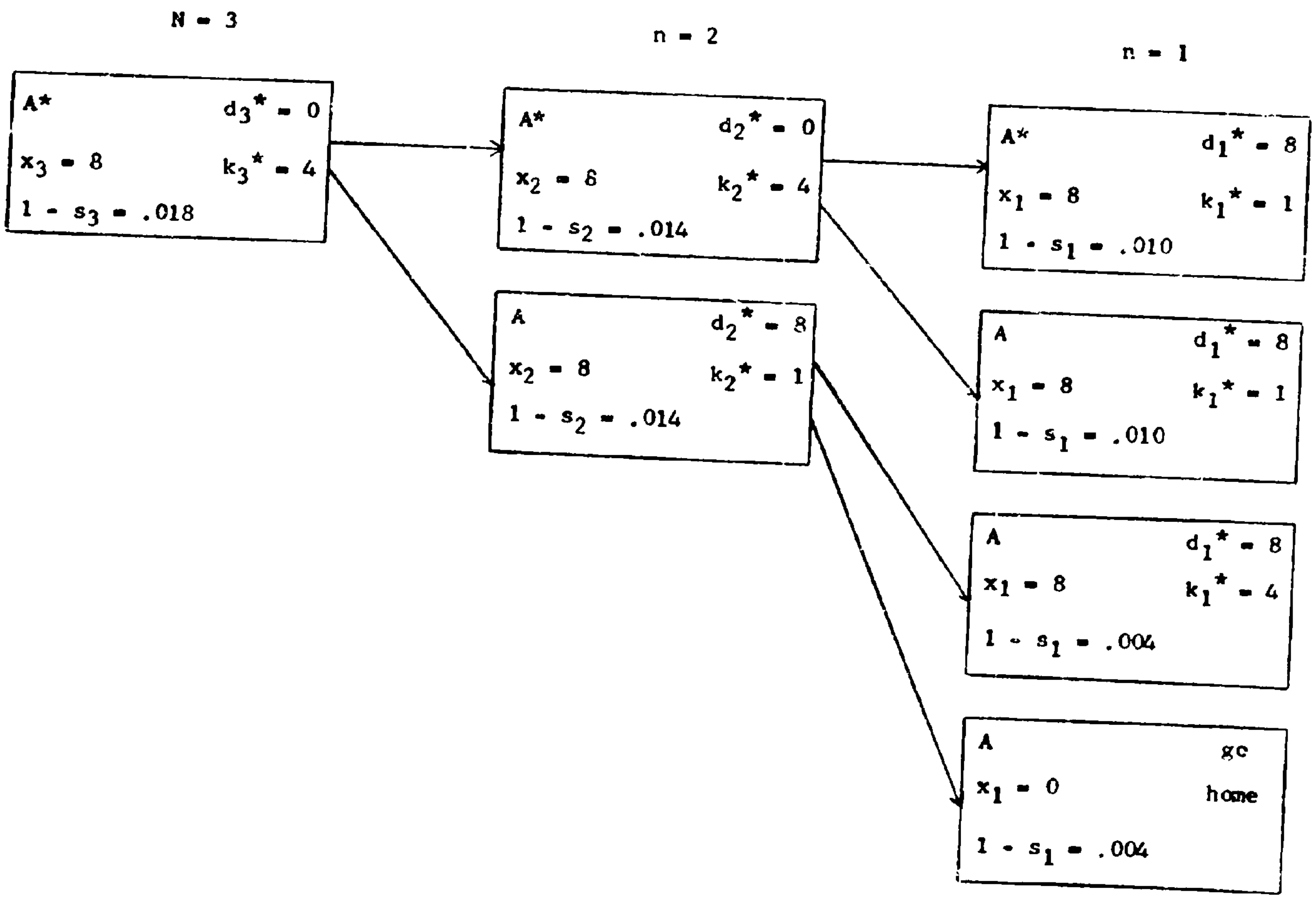


Fig. 16.--Optimum attack policy with reduced salvo effectiveness
$N = 3$; $1 - s_3 = .018$; $x_N = 8$; P_K duel

CHAPTER VI

GENERALIZING THE ATTACKER'S OBJECTIVE

An Aspiration Level Duel: The Simple "P_C Duel"

The P_K duel that was discussed in Chapter V is a special case of the duel that will be discussed next. It is possible that the attacking aircraft would wish to maximize the probability of obtaining at least C hits. This will be referred to as the "P_C duel." The aspiration level is C hits. The P_K duel is a special case of the P_C duel where the aspiration level is one hit.

First, consider a simple P_C duel which does not include probabilistic acquisition or multiple modes of attack. Let $h(\ell ; d_n)$ be the probability function of the number of hits, ℓ , per salvo of size d_n where ℓ is an integer such that $0 \leq \ell \leq d_n$.

The state of the system when preparing to make pass n can be described by specifying values for two state variables. The number of weapons remaining is x_n and since we are not for the moment including probabilistic target acquisition, d_n weapons are assumed to be delivered on pass n. The transformation of x_n is given by equation (1). For all $1 \leq n \leq N$,

$$x_{n-1} = x_n - d_n \tag{1}$$

Let i be the other state variable where $1 \leq i \leq I$. Let the number of hits already achieved be i-1.[1] Since the number of hits achieved on pass n is a random variable with a probability function that depends only on d_n, the transformation of i is probabilistic and we have a Markovian decision process. The Markov state transition probabilities can be stated as follows. Reference to Figure 17 will be helpful in following these relations. Note that $I = C + 1$ and for an arbitrary function of i, say $g(i)$, $\sum_{i=a}^{b} g(i) \equiv 0$, if $b < a$.

$$
\begin{aligned}
p_{ij}(d_n) &= h(j-i;\, d_n) && ; \quad 1 \leq i \leq I;\ i \leq j < I \\
&= \sum_{\ell=C-i+1}^{d_n} h(\ell;\, d_n) && ; \quad 1 \leq i < I;\ j = I \\
&= 1.0 && ; \quad i = j = I \\
&= 0 && ; \quad 1 \leq i \leq I;\ 1 \leq j < i \qquad (2)
\end{aligned}
$$

The transformation relationship equation (1) and the non negativity of d_n lead to the familiar sets

$$S_{x_n} = \left\{ x_n :\ x_n \in \left\{ 0, 1, \cdots, \bar{x}_N \right\} \right\} \qquad (3)$$

$$S_{d_n}(x_n) = \left\{ d_n :\ d_n \in \left\{ 0, 1, \cdots, x_n \right\} \right\} \qquad (4)$$

The return functions are rather simple. If one pass remains to be made, the probability of achieving at least C hits in the duel is

---

[1] We have set the state variable i equal to the number of hits plus one because we wish to allow for zero hits without destroying the convention that all index values start at one. This convention makes computer programming somewhat easier and lends consistency to the model.

	j = 1 0 hits	j = 2 1 hit	j = 3 2 hits		I = C + 1 # of hits $\geq$ C
i = 1 : 0 hits	$h(0;d_n)$	$h(1;d_n)$	$h(2;d_n)$	- - -	$\sum_{\ell=C}^{d_n} h(\ell;d_n)$
i = 2 : 1 hit	0	$h(0;d_n)$	$h(1;d_n)$	- - -	$\sum_{\ell=C-1}^{d_n} h(\ell;d_n)$
i = 3 : 2 hits	0	0	$h(0;d_n)$	- - -	$\sum_{\ell=C-2}^{d_n} h(\ell;d_n)$
I=C+1 : # of hits $\geq$ C	0	0	0	- - -	1.0

Fig. 17.--Transition probabilities for the simple P_C duel.

0.0 if transition is not to state I, is 1.0 if the system is already in state I, and is equal to the probability of surviving to the point of weapon delivery, S_T, if the transition is to state I from some other state. To summarize, if n = 1,

$$\begin{aligned} g_{1ij}\left[x_1, d_1, f_{o,j}(x_o)\right] &= 0.0 \quad ; \quad 1 \leq i < I; \quad 1 \leq j < I \\ &= 1.0 \quad ; \quad i = I \\ &= S_T \quad ; \quad 1 \leq i < I; \quad j = I \end{aligned} \tag{5}$$

When more than one pass remains to be made and transition is to some state other than I, the probability of achieving at least C hits in the duel is the probability of surviving pass n and achieving the remaining hits in the remaining n-1 passes. If the system is already in state I, the probability of achieving at least C hits in the duel is 1.0. If transition is to state I from some other state, the probability of achieving at least C hits in the duel is S_T. To summarize, if $2 \leq n \leq N$,

$$\begin{aligned} & g_{nij}\left[x_n, d_n, f_{n-1,j}(x_{n-1})\right] \\ &\quad = S_T S_u f_{n-1,j}(x_{n-1}) \quad ; \quad 1 \leq i < I \; ; \; 1 \leq j < I \\ &\quad = 1.0 \quad ; \quad i = I \\ &\quad = S_T \quad ; \quad 1 \leq i < I \; ; \; j = I \end{aligned} \tag{6}$$

We have now defined all of the parts of the basic functional equation for the Markovian decision process, equation (II-18). Substituting the expressions from equation (2) for the $p_{ij}(d_n)$ term,

substituting the expressions from equations (5) and (6) for the $g_{nij}\left[\cdots\right]$ term and using the sets defined by equations (3) and (4) gives the following. If $i = I$ and for all $1 \leq n \leq N$

$$f_{nI}(x_n) = 1.0 \tag{7}$$

If $n = 1$ and $1 \leq i < I$,

$$f_{1i}(x_1) = \max_{0 \leq d_1 \leq x_1} S_T \sum_{\ell=C-i+1}^{d_1} h(\ell;d_1) \tag{8}$$

If $2 \leq n \leq N$ and $1 \leq i < I$,

$$f_{ni}(x_n) = \max_{0 \leq d_n \leq x_n} \left\{ \sum_{j=i}^{C} h(j-i;d_n)S_T S_u f_{n-1,j}(x_{n-1}) + S_T \sum_{\ell=C-i+1}^{d_n} h(\ell;d_n) \right. \tag{9}$$

Since the return functions, equations (5) and (6) clearly satisfy the monotonicity and equivalence conditions of Chapter II, recursive application of equations (7), (8), and (9) will yield the maximum n stage return and the optimum weapon delivery policy for the simple P_C duel.

Note that the optimum policy indicates the best act as a function of n, i, and x_n. To implement this policy, the pilot must know the number of hits already achieved. The implications of this will be discussed later.

A Generalized P_C Duel

In generalizing the model for the P_C duel to include probabilistic acquisition and multiple modes of attack, let $h(\ell;D_n) = h(\ell;d_n,k_n)$

be the probability function of the number of hits, ℓ, achieved on pass n when d_n is the salvo size and mode of attack k_n is selected. The maximum n stage return, $f_{ni}(X_n') = f_{ni}(x_n, s_n)$ is the maximum probability of achieving at least C hits in the duel when n passes are yet to be made, x_n weapons remain, the probability of surviving the remaining n passes must be at least s_n, and the system is in Markov state i.

The Markov state must now reflect both the number of hits that have already been achieved and the acquisition status of the system. One possible approach would be the definition of a two dimensional vector to characterize the Markov state of e system. We will take the approach, however, of defining Markov states in such a way that a single dimension Markov-state variable can reflect both the number of hits already achieved and the acquisition status of the system. The functional equation (II-18) can then be used directly to optimize the return. To accomplish this, the Markov states are as defined in Table 16.

Figure 18 shows the transition probabilities for the case where C = 3. Using this figure as a guide, we can construct the transition probabilities for the general case. If i = 1,

$$
\begin{aligned}
p_{ij}(D_n) &= 1 - P(A_o) && ;\ j = 1 \\
&= P(A_o) - P(A_oD) && ;\ j = 2 \\
&= 0 && ;\ 3 \le j \le C+1 \\
&= P(A_oD)\ h(j-C-2;\ D_n) && ;\ C+2 \le j \le 2C+1 \\
&= P(A_oD) \sum_{\ell=C}^{d_n} h(\ell;D_n) && ;\ j = I = 2C+2 \qquad (10)
\end{aligned}
$$

TABLE 16

DEFINITIONS OF MARKOV STATES IN THE GENERALIZED P_C DUEL

State Variable	Definition
i = 1	A* = Acquisition has not occurred
i = 2	AD*;0 hits = Acquisition has occurred, delivery did not occur on the most recent pass, and no hits have been scored
i = 3	AD*;1 hit = same as above with 1 hit scored
i = 4	AD*;2 hits = same as above with 2 hits scored
.	
.	
.	
i = C + 1	AD*;C-1 hits = same as above with C-1 hits scored
i = C + 2	AD;0 hits = Acquisition has occurred, delivery occurred on the most recent pass, and no hits have been achieved
i = C + 3	AD;1 hit = same as above with 1 hit scored
i = C + 4	AD;2 hits = same as above with 2 hits scored
.	
.	
.	
i = 2C + 1	AD;C-1 hits = same as above with C-1 hits scored
i = I = 2C + 2	At least C hits have been achieved

	j=1 A*	j=2 AD*; 0 hits	j=3 AD*; 1 hit	j=4 AD*; 2 hits	j=5 AD; 0 hits	j=6 AD; 1 hit	j=7 AD; 2 hits	j=I=2C+2 at least C hits
A*	$1-P(A_o)$	$P(A_o)-P(A_oD)$	0	0	$P(A_oD)h(0;D_n)$	$P(A_oD)h(1;D_n)$	$P(A_oD)h(2;D_n)$	$P(A_oD)\sum_{\ell=C}^{d_n} h(\ell;D_n)$
AD* 0 hits	0	$1-P(A_1D)$	0	0	$P(A_1D)h(0;D_n)$	$P(A_1D)h(1;D_n)$	$P(A_1D)h(2;D_n)$	$P(A_1D)\sum_{\ell=C}^{d_n} h(\ell;D_n)$
AD* 1 hit	0	0	$1-P(A_1D)$	0	0	$P(A_1D)h(0;D_n)$	$P(A_1D)h(1;D_n)$	$P(A_1D)\sum_{\ell=C-1}^{d_n} h(\ell;D_n)$
AD* 2 hits	0	0	0	$1-P(A_1D)$	0	0	$P(A_1D)h(0;D_n)$	$P(A_1D)\sum_{\ell=C-2}^{d_n} h(\ell;D_n)$
AD 0 hits	0	$1-P(A_1D)$	0	0	$P(A_1D)h(0;D_n)$	$P(A_1D)h(1;D_n)$	$P(A_1D)h(2;D_n)$	$P(A_1D)\sum_{\ell=C}^{d_n} h(\ell;D_n)$
AD 1 hit	0	0	$1-P(A_1D)$	0	0	$P(A_1D)h(0;D_n)$	$P(A_1D)h(1;D_n)$	$P(A_1D)\sum_{\ell=C-1}^{d_n} h(\ell;D_n)$
AD 2 hits	0	0	0	$1-P(A_1D)$	0	0	$P(A_1D)h(0;D_n)$	$P(A_1D)\sum_{\ell=C-2}^{d_n} h(\ell;D_n)$
at least C hits	0	0	0	0	0	0	0	1.0

Fig. 18.--Transition probabilities for the general P_C duel when C = 3.

If $2 \leq i \leq C+1$,

$$
\begin{aligned}
p_{ij}(D_n) &= 0 && ;\ 1 \leq j < i;\ i < j < C+1 \\
&= 1 - P(A_1 D) && ;\ j = i \\
&= P(A_1 D)\ h(j-C-i,\ D_n) && ;\ C+1 \leq j \leq 2C+1 \\
&= P(A_1 D) \sum_{\ell=C-i+2}^{d_n} h(\ ; D_n) && ;\ j = I = 2C+2 \qquad (11)
\end{aligned}
$$

If $C+2 \leq i \leq 2C+1$,

$$
p_{ij}(D_n) = p_{i-C,j}(D_n) \qquad ;\ 1 \leq j \leq 2C+1 \qquad (12)
$$

If $i = I = 2C+2$,

$$
\begin{aligned}
p_{ij}(D_n) &= 0 && ;\ 1 \leq j \leq 2C+1 \\
&= 1.0 && ;\ j = I = 2C+2 \qquad (13)
\end{aligned}
$$

The residual state variable transformation relations are as follows. For all $1 \leq n \leq N$ and $1 \leq i \leq 2C+2$,

$$
\begin{aligned}
x_{n-1} &= x_n && ;\ 1 \leq j \leq C+1 \\
&= x_n - d_n && ;\ C+2 \leq j \leq 2C+2 \qquad (14)
\end{aligned}
$$

$$
s_{n-1} = \frac{s_n}{S_T(k_n) S_u(k_n)} \qquad ;\ \text{all } j \qquad (15)
$$

In the transformation equation (15) we are ignoring the problem that in making computations, s_n must take discrete values. The above transformation equations lead to the sets defined by equations (IV-21), (IV-22), (IV-23), and (IV-24).

In defining the return functions, we get some compensation for the complexity of the transition probabilities. The return functions are identical to equations (5) and (6) for the simple P_C duel except that x_n is replaced by $X_n' = (x_n, s_n)$, d_n is replaced by $D_n = (d_n, k_n)$, S_T and S_u are now $S_T(k_n)$ and $S_u(k_n)$, and $I = 2C+2$.

All the parts of the basic functional equation (II-18) have now been defined for the P_C duel with acquisition and multiple modes of attack.

The Simple Expected Damage Duel: The Simple "E_D Duel"

The aspiration level objective of the P_C duel allows no utility for any number of hits less than C and no marginal utility for additional hits once C hits have been obtained. This is a tenable abstraction for some situations where $C = 1$, i.e., the P_K duel. When $C > 1$, not very many applications come to mind. Almost any situation requiring three hits offers some value for two hits. The P_C duel is included because it fits logically into the pattern of duels that are considered and it makes a convenient way of introducing the duel that is to be considered next.

This leads to the final generalization of the duel objective. The E_H duel, the P_K duel, and the P_C duel are all special cases. Suppose a utility or damage level is associated with the number of hits scored. If the Markov state variable i reflects the number of hits achieved, then a function U(i) can associate the damage level achieved with the Markov state of the system. The damage level achieved on a

sortie is a random variable. Maximizing its expected value is a reasonable objective. This will be referred to as maximizing expected damage. The present case will be called the E_D duel.

The functional equations for optimizing the simple E_D duel can be obtained by simple modifications of the equations for the simple P_C duel that was discussed in the first part of this chapter.

In this development, $f_{ni}(x_n)$ is the maximum expected value of the marginal (or additional) damage achievable in the remaining n passes when x_n weapons remain and the system is in Markov state i. The quantity $h(\ell;d_n)$ is still the probability function of the number of hits, ℓ, per salvo of size d_n where $0 \le \ell \le d_n$.

The transformation relation for x_n is given by equation (1). The Markov state variable i has the same definition it did in the simple P_C duel (number of hits achieved = i - 1). The Markov state transition probabilities are given by equation (2) and Figure 17. The quantity C is reinterpreted as the number of hits associated with the maximum damage level, i.e., additional hits are of no further value. The sets S_{x_n} and $S_{d_n}(x_n)$ are defined by equations (3) and (4).

In general terms, the composition of the stage n return with the maximum return obtainable from the remaining n-1 stages is the discounted sum of the marginal damage achievable on pass n and the additional damage achievable on the remaining n-1 passes starting from the state that results from pass n. More specifically, if n = 1 and $1 \le i < I$ (reference to Figure 17 may be helpful),

$$g_{1ij}\left[x_1, d_1, f_{o,j}(x_o)\right] = S_T\left[U(j) - U(i)\right] \qquad ; i \le j \le I \qquad (16)$$

If $n = 1$ and $i = I$,

$$g_{1ij}\left[x_1, d_1, f_{o,j}(x_o)\right] = 0 \qquad ; \ j = I \qquad (17)$$

since damage level saturation has already been reached before making the pass.

If $2 \leq n \leq N$ and $1 \leq i < I$,

$$g_{nij}\left[x_n, d_n, f_{n-1,j}(x_{n-1})\right]$$

$$= S_T\left[U(j) - U(i)\right] + S_T S_u f_{n-1,j}(x_{n-1}) \quad ; \quad i \leq j < I$$

$$= S_T\left[U(I) - U(i)\right] \qquad ; \quad j = I \qquad (18)$$

If $2 \leq n \leq N$ and $i = I$,

$$g_{nij}\left[x_n, d_n, f_{n-1,j}(x_{n-1})\right] = 0 \qquad ; \quad j = I \qquad (19)$$

Note that in equations (16), (17), (18), and (19), the return functions are not defined for the cases where $j < i$. This is justified because $p_{ij}(d_n) = 0$ if $j < i$.

All of the required parts of equation (II-18) have now been defined so the solution for the simple E_D duel can be obtained.

The General E_D Duel

To include acquisition and multiple modes of attack in the E_D duel, $h(\ell; d_n)$ becomes $h(\ell; D_n)$ and $f_{ni}(x_n)$ becomes $f_{ni}(X_n')$. $U(i)$ is the damage level associated with Markov state i. The Markov states are defined in Table 16. The Markov state transition probabilities are given by equations (10), (11), (12) and (13). The residual state variable transformations are given by equations (14) and

(15). Appropriate sets are defined in equations (IV-21), (IV-22), (IV-23) and (IV-24).

The return functions for this case can be constructed as follows. The quantity C must be interpreted as the number of hits associated with the damage saturation level, i.e., achieving more than C hits is no better than achieving C hits.

If one pass remains, the expected return for given X_1', D_1, i, and j is simply the probability of surviving to the point of weapon delivery times the difference in damage level associated with the Markov states i and j. Thus, if $n = 1$ and for all $1 \leq i \leq 2C+2$,

$$g_{1ij}\left[X_1', D_1, f_{o,j}(X_o')\right] = S_T(k_1)\left[U(j) - U(i)\right] \; ; \; 1 \leq j \leq 2C+2 \tag{20}$$

When more than one pass remains, the expected return for given X_n', D_n, i and j is the probability of surviving to the point of weapon release on pass n times the difference in damage level associated with the Markov states i and j plus the probability of surviving pass n times the additional damage achievable on the remaining n - 1 passes. Symbolically, if $2 \leq n \leq N$ and for all $1 \leq i \leq 2C+2$,

$$g_{nij}\left[X_i', D_i, f_{n-1,j}(X_{n-1}')\right] =$$

$$S_T(k_n)\left[U(j)-U(i)\right] + S_T(k_n)S_u(k_n)f_{n-1,j}(X_{n-1}') \; ;$$

$$1 \leq j \leq 2C+2 \tag{21}$$

The case where $U(j) < U(i)$ will occur among the array of combinations of i and j that are covered by equations (20) and (21).

This case does not create a problem because the transition probabilities, $p_{ij}(k_n)$, associated with all such cases are zero. This can be verified by examining Figure 18.

All of the parts of equation (II-18) have now been defined so the maximum expected damage and the optimum policy can be determined for the general E_D duel. The optimal policy tells the pilot how many weapons to deliver and what mode of attack to adopt depending on the number of passes remaining, the number of weapons remaining, the acquisition status, and the number of hits that have been achieved.

The existence of real situations in which the pilot knows exactly how many hits have been scored is debatable. Conversations with experienced pilots indicate that the pilot generally does not know how many hits have been scored but he is not completely ignorant of the effectiveness of his passes. He may be able to watch the effect during pullout or a fellow pilot may make observations. Further, whether or not the target can be observed, the pilot has some idea of whether he has made a good delivery. The ability of the pilot to estimate salvo effectiveness is highly variable depending on the conditions of the attack and the nature of the target.

In some cases it may be appropriate to act as if the number of hits scored or current damage level at each stage is known. In such cases, the previously discussed methods will yield optimal tactics.

Implications of Unobservable Markov State Transitions

It is interesting to see what is involved under the assumption that the pilot has no information at all about the effectiveness of his

previous passes. The simple P_C duel will serve as a vehicle for examining this matter.

When the Markov state transitions are unobservable, the simple P_C duel becomes a Markov decision process with unobservable transitions. Such a process was discussed in Chapter II. The probability function for the Markov state of the system, $\pi_n(i; \pi_N, d_N, \text{---}, d_{n+1})$, can be determined by recursive application of the relation

$$\pi_n(j;\pi_N,d_N,\text{---},d_{n+1}) = \sum_{i=1}^{I} \pi_{n+1}(i;\pi_N,d_N,\text{---},d_{n+2})p_{ij}(d_n) \tag{22}$$

where the $p_{ij}(d_n)$ are given by equation (2).

The functional equation for this duel can be developed from equations (8) and (9) for the simple P_C duel by the same argument that was used to develop equation (II-21) by starting with equation (II-18). We can apply this argument as follows. Since the Markov state, i, of the system is only known probabilistically at each stage, the maximum n stage return is the maximum expected value where the expectation is taken over the random variable i. Since this maximum n stage return depends on the initial probability function, π_N, and it depends on the decisions $D_N,\text{---},D_{n+1}$, it is denoted $f_n(X_n', \pi_N, D_N,\text{---},D_{n+1})$. The return function is indicated by $g_{nij}\left[X_n', D_n, f_{n-1}(X_{n-1}', \pi_N,D_N,\text{---},D_n)\right]$ and in the present application to the simple P_C duel, the return functions are similar to equations (5) and (6). If n = 1,

$$\begin{aligned} g_{1ij}\left[X_1,d_1,f_0(x_0,\pi_N,d_N,\text{---},d_1)\right] &= 0.0 \;; \quad 1 \le i < I;\ 1 \le j < I \\ &= 1.0 \;; \quad i = I \\ &= S_T \;; \quad 1 \le i < I;\ j = I \end{aligned} \tag{23}$$

If $2 \le n \le N$,

$$g_{nij}\left[x_n, d_n, f_{n-1}(x_{n-1}, \pi_N, d_N, ---, d_n)\right]$$

$$\begin{aligned} &= S_T S_u f_{n-1}(x_{n-1}, \pi_N, d_N, ---, d_n) && ; 1 \le i < I;\ 1 \le j < I \\ &= 1.0 && ; i = I \\ &= S_T && ; 1 \le i < I;\ j = I \quad (24) \end{aligned}$$

Now, substituting the Markov state probability functions from equation (22), the Markov state transition probabilities from equation (2), and the return functions from equations (23) and (24) into equation (II-21) gives the following functional equations for the P_C duel with unobservable Markov state transitions.

If $n = 1$,

$$f_1(x_1, \pi_N, d_N, ---d_2) = \max_{0 \le d_1 \le x_1} \sum_{i=1}^{I} \pi_1(i; \pi_N, d_N, ---, d_2) S_T \sum_{\ell=C-i+1}^{d_1} h(\ell; d_1) \quad (25)$$

If $2 \le n \le N$

$$f_n(x_n, \pi_N, d_N, ---d_{n+1}) = \max_{0 \le d_n \le x_n} \sum_{i=1}^{I} \pi_n(i; \pi_N, d_N, ---, d_{n+1}) \left[\sum_{j=1}^{C} h(j-i; d_n) S_T S_u f_{n-1}(x_{n-1}, \pi_N, d_N, ---, d_n) + S_T \sum_{\ell=C-i+1}^{d_n} h(\ell; d_n) \right] \quad (26)$$

As indicated in Chapter II, the high dimensionality seems to make equations (25) and (26) impractical to implement.

The prospective difficulty of implementing equations (25) and (26) illustrates a far reaching difficulty in the study of military duels and in studies of many other areas. The solution methods for Markovian decision processes that are discussed in this work and by Howard (17), the solution methods for stochastic games discussed by Charnes and Schroeder (11) and by Shapley (23), and other related solutions all provide an optimal policy or strategy that indicates how to act as a function of the state of the system. This always presupposes a perfect knowledge of the state of the system on the part of the actor. The complications that we have faced in this section are indicative of the problems that arise when perfect knowledge of the state of the system can not be assumed.

CHAPTER VII

MULTIPLE AIRCRAFT RAIDS

General Considerations

Discussion in previous chapters has involved the duel between a single aircraft and a defended target. This duel has been characterized by the return-versus-attrition function. Such functions were computed in previous chapters for examples of the simple E_H duel (Figure 5), the general E_H duel (Figure 11) and the general P_K duel (Figure 14). A return-versus-attrition function along with the statement of optimal attack policy at each attrition level is informative, but is generally not adequate for decision making. It offers no indication of which attrition level should be adopted. The purpose of this chapter is to shed some light on the selection of an attrition level for the duel, i.e., selection of the aircraft's attack policy.

The return-versus-attrition function represents a tradeoff between effectiveness and cost. Our approach will be to minimize the cost of achieving a given level of effectiveness. In this report, cost is in terms of expected aircraft losses. The units of return depend on the situation.

A raid is visualized as follows. A group of R aircraft departs from its base and penetrates enemy area defenses to the vicinity of the target. Each aircraft attacks the target according to a predetermined policy. When the attack is completed, the aircraft penetrate

enemy area defenses and return to their base. The [illegible] problems include specifying the raid size R and the attack p[illegible] that each aircraft will follow.

Using the E_H Duel as a Basis

Objective

A raid model based on the expected value maximization of Chapters III and IV would minimize the expected losses to achieve a given expected value of the number of hits. This type of model would be useful if the damage level or utility is a linear function of the number of hits obtained.

Note that not all hits need be on the same object. An example target where hits are not all on the same object is a dispersed supply depot consisting of many small supply caches defended by a common defense system. If the aircraft were to attack a different supply cache on each pass and if $r_H(d_n)$ is interpreted as the expected hits per salvo, then $f_N(X_N)$ would be the maximum expected hits per duel. Note that $r_H(d_n)$ might also be interpreted as the probability of killing a supply cache versus salvo size in which case $f_N(X_N)$ would be the maximum expected caches killed per duel.

The Return-Versus-Attrition Function

In considering the return-versus-attrition function that is produced by the models of the previous chapters, the choice is among a number of different attack policies which will be indexed $1 \leq m \leq M$. Each attack policy corresponds to one of the values of the constraining probability of the aircraft surviving the duel, s_N, where $s_N \in S_{s_N}$.

The return for the generalized E_H duel, $f_{N1}(x_N,s_N)$, is the maximum expected hits if we assume that at the beginning of the duel, target acquisition has not yet occurred, i.e., $i = 1$. For notational convenience, we will denote this return as $u_D(m)$. The attrition factor will be symbolized in terms of the probability of surviving the duel. Let $S_D(m)$ be the probability of the aircraft surviving the duel when policy m is selected.

Note that $S_D(m)$ is the actual probability of surviving the duel under the m^{th} attack policy. This may differ from the corresponding constraining value s_N. Accordingly, the first task is to modify the return-versus-attrition function to reflect actual probability of survival rather than constraining values. To accomplish this, each of the M attack policies is evaluated to determine the resulting value of $S_D(m)$. It is convenient to perform this evaluation by using recursive techniques.

For the generalized E_H duel, the optimal policy is given by $d_{ni}^*(x_n,s_n)$ and $k_{ni}^*(x_n,s_n)$, to be abbreviated d_n and k_n, respectively. Let $\rho_{ni}(x_n,s_n)$ be the actual probability of survival when n passes remain, the system is in Markov state i, x_n weapons remain, the constraining probability of survival is s_n, and the corresponding optimal policy is followed. The actual probability of survival can be evaluated recursively by using the following relationships.

If $n = 1$,

$$\rho_{1i}(x_1,s_1) = S_T(k_1^*)\ S_u(k_1^*) \tag{1}$$

If $2 \leq n \leq N$,

$$\rho_{ni}(x_n,s_n) = \sum_{j=1}^{3} p_{ij}(d_n^*)\ S_T(k_n^*)\ S_u(k_n^*)\ \rho_{n-1,j}(x_{n-1},s_{n-1}) \tag{2}$$

where x_{n-1} is given by transformation equations (IV-15) and s_{n-1} is given by the discrete version of the survival constraint transformation equation (IV-34).

If the appropriate m is associated with each s_N, then for given N and x_N, $\phi_{N1}(x_N, s_N) = S_D(m)$. Note that at pass N, i.e., the first pass, the system is always assumed to be in Markov state 1.

<u>A Model for Minimizing Expected Losses</u>

Suppose that the aircraft in the raid make stochastically independent, statistically identical attacks against the target. Let S_A be the probability that a given aircraft survives the area defenses from its base to the target and suppose that the probability of surviving area defenses is the same on the return from the target to the base. The raid size required to realize C_R expected hits is given by

$$R(m) = \frac{C_R}{u_D(m)\ S_A} \tag{3}$$

where C_R expresses the desired level of accomplishment.[1] The probability that a given aircraft does not survive the raid is given by

$$1 - S_A^2\ S_D(m)$$

Thus, the expected value of the number of aircraft lost per raid is given by

$$E_R(m) = \frac{C_R}{u_D(m)\ S_A}\ (1 - S_A^2\ S_D(m)) \tag{4}$$

---

[1]In making numerical calculations, R(m) should take the smallest integer value no smaller than the value of the right side of equation (3). This is important when that value is small, especially if it is less than 1.0. We have ignored this here.

The minimum expected value of the number of aircraft lost per raid is given by

$$L_R = \frac{C_R}{S_A}\left\{ \min_{1 \leq m \leq M} \left[\frac{1-S_A{}^2 S_D(m)}{u_D(m)} \right] \right\} \tag{5}$$

The minimizing value of m will be m* and R* = R(m*) will be the optimum raid size.

Key assumptions of this raid model are as follows:

a. The same probability of surviving the area defenses applies enroute from the base to the target and returning from the target to the base.

b. The aircraft in the raid fly stochastically independent, statistically identical sorties.

c. At least R* aircraft are available.

Assumption a. could easily be relaxed but doing so would only add to the complexity of this work without adding substantially to its content. To relax assumption a., we would simply distinguish between the probability of surviving from the base to the target and the probability of surviving the return flight. The effect on the equations in the model would be minor and they could easily be modified to reflect the change.

Assumption b. is the most important one since it is a principle basis of the raid model. That assumption implies that all aircraft duels are characterized by the same functions $S_T(k_n)$, $S_u(k_n)$, $r(d_n, k_n)$, and $p_{ij}(k_n)$. As was pointed out in the first section of Chapter III, the return functions are indexed according to stage, n, which means

that the foregoing functions could be stage dependent without violating assumption b. as long as each aircraft's duel is characterized by the same set of functions. This is important because the survival probability functions, $S_T(k_n)$ and $S_u(k_n)$, generally do depend on n. Conversations with combat pilots indicate that attrition on the first pass is generally much lower than attrition on later passes because of the surprise element. This dependence can be reflected by making the functions $S_T(k_n)$ and $S_u(k_n)$ depend on n in the recursive calculations. It is useful to be able to reflect this dependence without invalidating the raid model.

The notion that all aircraft might make statistically identical attacks is reinforced by the following. When a group of aircraft attack a target, the attack is often arranged so all the aircraft make their first pass within a short period of time, i.e., each aircraft gets the advantage of surprise on its first pass. If a second pass is intended, then it seems reasonable to assume that the surprise element is no longer present for any of the aircraft.

A situation in which the aircraft attacks can be assumed to be stochastically independent and statistically identical arises in the all-weather operation of newer weapon systems. Since these aircraft are designed to make attacks under bad weather conditions, coordination of the attack is difficult and therefore in designing weapon systems and planning attacks, it might be assumed that when multiple aircraft are involved in a raid, they operate independently.

Assumption c. is implied by the fact that raid size is selected and the policy for the duel is determined without regard for the number

of aircraft available. If the selected raid size, R^*, exceeds the number of aircraft available, the objective of the raid must be relaxed. In a raid based on the E_H duel, this can be accomplished by reducing the value of C_R.

Numerical Example

The example of Chapter IV illustrates the application of the model for the E_H duel. Tables 5 and 6 illustrate the $d_{ni}^*(x_n,s_n)$ and $k_{ni}^*(x_n,s_n)$, respectively, that resulted in that example and Figure 11 shows the return-versus-attrition function with the attrition in terms of attrition constraint. Figure 19 shows a duplicate of the return-versus-attrition function from Figure 11 along with the modified function that results when equations (1) and (2) are used to determine the actual attrition associated with each point.

Suppose a raid is to be planned so as to minimize the expected losses incurred in realizing an expected value of the number of hits, C_R, equal to 10.0. Suppose $S_A = 0.995$. Applying the raid model of equation (5) to the $u_D(m)$ versus $[1 - S_D(m)]$ function shown in Figure 19, the expected value of the number of aircraft lost is minimized by choosing the minimum point on the $E_R(m)$ function shown in Figure 20. The minimum point occurs at the expected attrition level $1 - S_D(m) =$ .0083. This corresponds to a constraining expected attrition value of $1 - s_N = 0.012$. The optimum attack policy for this expected attrition level is diagrammed in Figure 10 and was discussed in Chapter IV. The optimal raid size given by equation (3) for this attrition level is 11.8 aircraft. The expected losses per raid is $L_R = 0.215$ aircraft.

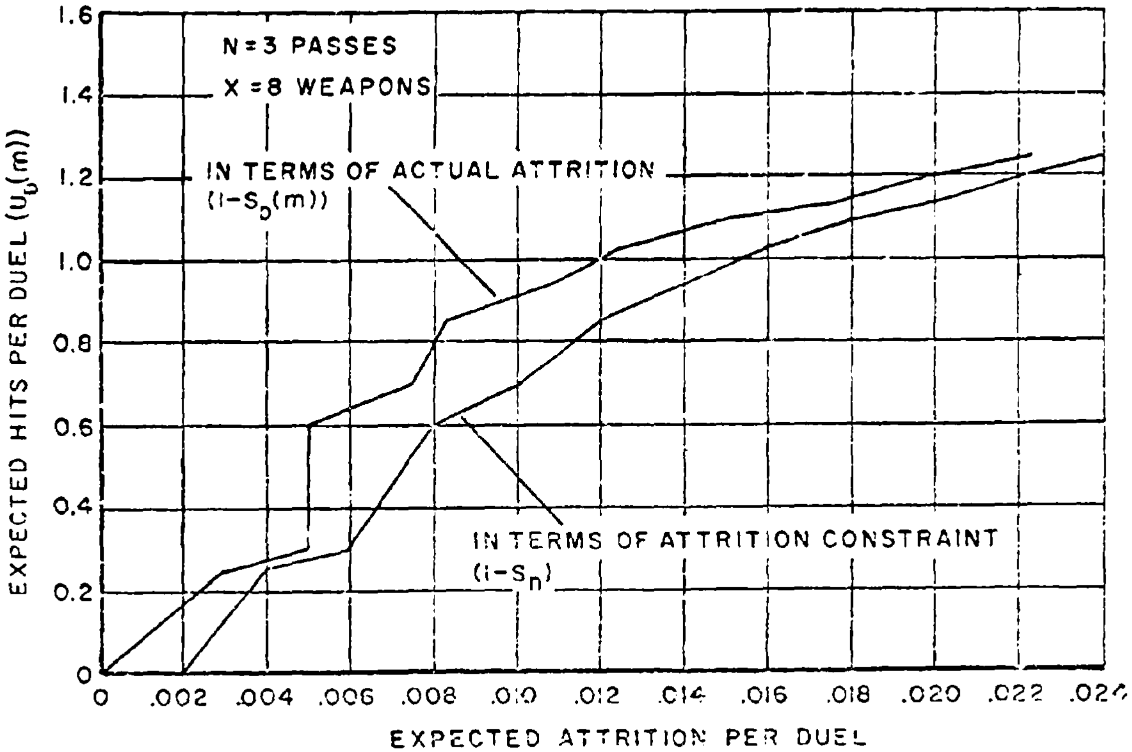


Fig. 19.--Modified return-versus-attrition function for the E_H duel.

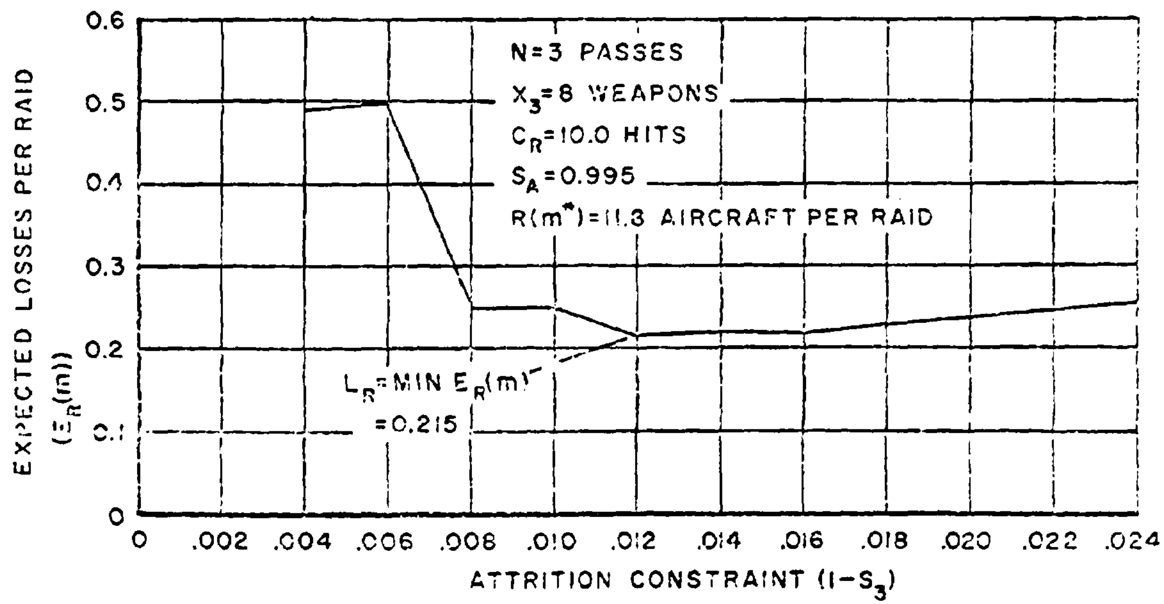


Fig. 20.--Minimizing expected losses per raid in the E_H duel.

Using the P_K Duel as a Basis

The Return-Versus-Attrition Function

A raid model based on the P_K duel can also be developed. In the P_K duel the quantity $f_{N1}(x_N, s_N)$ represents the maximum probability of achieving at least one hit given that the aircraft survives the area defenses from the base to the target. That probability will be represented here by $K_D(m)$. The modified return-versus-attrition function for this situation relates $K_D(m)$ to the probability of not surviving the duel, $1 - S_D(m)$.

A Model for Minimizing Expected Losses

Suppose raid size, R^*, and attack policy, m^*, are to be selected so as to minimize the expected losses incurred in achieving a probability K_R of getting at least one hit during the raid.

If $R(m)$ aircraft make stochastically independent, statistically identical attacks using policy m, the probability that none of the aircraft gets a hit is

$$(1 - S_A K_D(m))^{R(m)} = 1 - K_R \tag{6}$$

The raid size required to realize K_R is therefore[2]

$$R(m) = \frac{\ln(1 - K_R)}{\ln(1 - S_A K_D(m))} \tag{7}$$

---

[2]In making numerical calculations, R(m) should take the smallest integer value no smaller than the value of the right side of equation (7). This is important when that value is small, especially if it is less than 1.0. We have ignored this here.

Since the probability of a given aircraft not surviving is given by

$$1 - S_A^2\ S_D(m)$$

the expected value of the number of aircraft lost per raid is

$$E_R(m) = \left[1 - S_A^2\ S_D(m)\right]\ R(m)$$

$$= \left[1 - S_A^2\ S_D(m)\right] \frac{\ln(1 - K_R)}{\ln(1 - S_A\ K_D(m))} \tag{8}$$

Thus, the minimum expected losses per raid is

$$L_R = \ln(1 - K_R) \left\{ \underset{1 \leq m \leq M}{\mathrm{Min}} \left[\frac{1 - S_A^2\ S_D(m)}{1 - S_A\ K_D(m)} \right] \right\} \tag{9}$$

It might be assumed that if the kill is not achieved on one raid, another raid will have to be undertaken and that raids will be repeated until the job is finally done. Suppose all the raids are to be stochastically identical and let target kill occur on the ith raid. Then the expected value of the number of aircraft lost in killing the target is

$$E_T(m) = \sum_{i=1}^{\infty} E_R(m)\ i\ K_R(1 - K_R)^{i-1}$$

$$= E_R(m)\ K_R \sum_{i=0}^{\infty} (i+1)(1-K_R)^i = \frac{E_R(m)}{K_R} \tag{10}$$

Thus, the minimum expected losses to kill the target is

$$L_T = \frac{1}{K_R} \left\{ \underset{1 \leq m \leq M}{\mathrm{Min}} \quad E_R(m) \right\} = \frac{L_R}{K_R} \tag{11}$$

So, under these assumptions, the raid size and attack policy that minimizes expected losses per raid also minimizes expected losses incurred in finally killing the target.

Key assumptions for this raid can be summarized as follows:

a. The same probability of surviving the area defenses applies enroute from the base to the target and returning from the target to the base.

b. The aircraft in the raid make stochastically independent, statistically identical attacks.

c. At least R* aircraft are available.

d. Statistically identical raids are repeated until the target is killed at which time the raids cease. This assumption only applies when computing L_T.

Assumptions a. and b. are the same as the first two assumptions listed in the previous section and the same comments apply. Regarding assumption c., if the selected raid size exceeds the number of aircraft available, the value of K_R must be reduced.

Numerical Example

The example of Chapter V illustrates the application of the recursive equations for the P_K duel and results in the return-versus-attrition function that is given in Figure 14. Figure 21 shows that same return-versus-attrition function along with the corresponding modified function that is obtained by using equations (1) and (2). This modified return-versus-attrition function indicates the maximum probability of killing the target versus the actual probability of

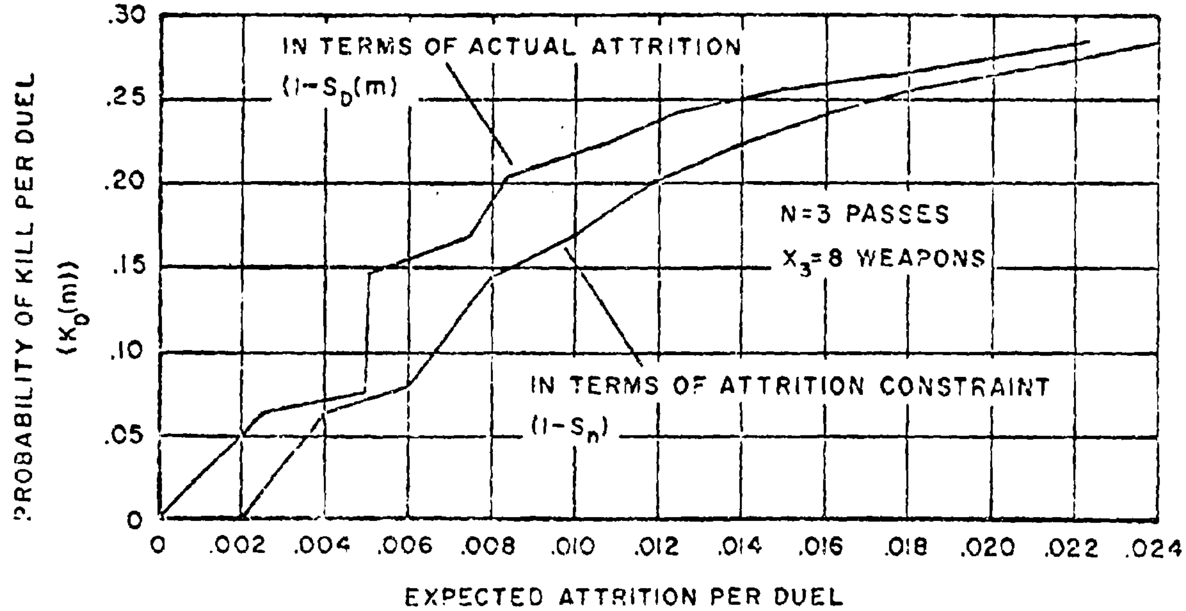


Fig. 21.--Modified return versus attrition function for the P_K duel.

the aircraft surviving the duel. Recall from Chapter V that this function applies when three passes and eight weapons are available and target acquisition has not yet occurred.

Suppose a raid is to be planned so as to minimize the expected losses incurred in realizing a probability of $K_R = 0.9$ of getting at least one hit. Suppose $S_A = 0.995$. Application of the raid model of equation (9) to the $K_D(m)$ versus $[1 - S_D(m)]$ function plotted in Figure 21 is illustrated in Figure 22. The minimum loss point occurs at the attrition level $1 - S_D(m) = 0.0083$ which corresponds to an attrition constraint value of $1 - s_N = 0.012$. The optimal raid size given by equation (7) for this attrition level is 10.3 aircraft. The minimum expected losses per raid is $L_R = 0.187$. The optimum attack policy for this attrition level is diagrammed in Figure 13.

Multiple Target Raids

The raid model to be developed next visualizes an operational planner who has a given number of identical aircraft available to be dispatched simultaneously on air-to-ground attack sorties. He has available an array of targets that are of varying difficulty and value. The planner must decide how many sorties to allocate to each target, and he must designate an attack policy to be used for each of the raids. He must deal with a tradeoff between total utility achieved and expected losses. It is our intention here to use the individual duel results as a basis for studying this problem.

First, consider a generalized raid model that is based on the E_H duel. Suppose there are T targets available indexed $t = 1,---,T$.

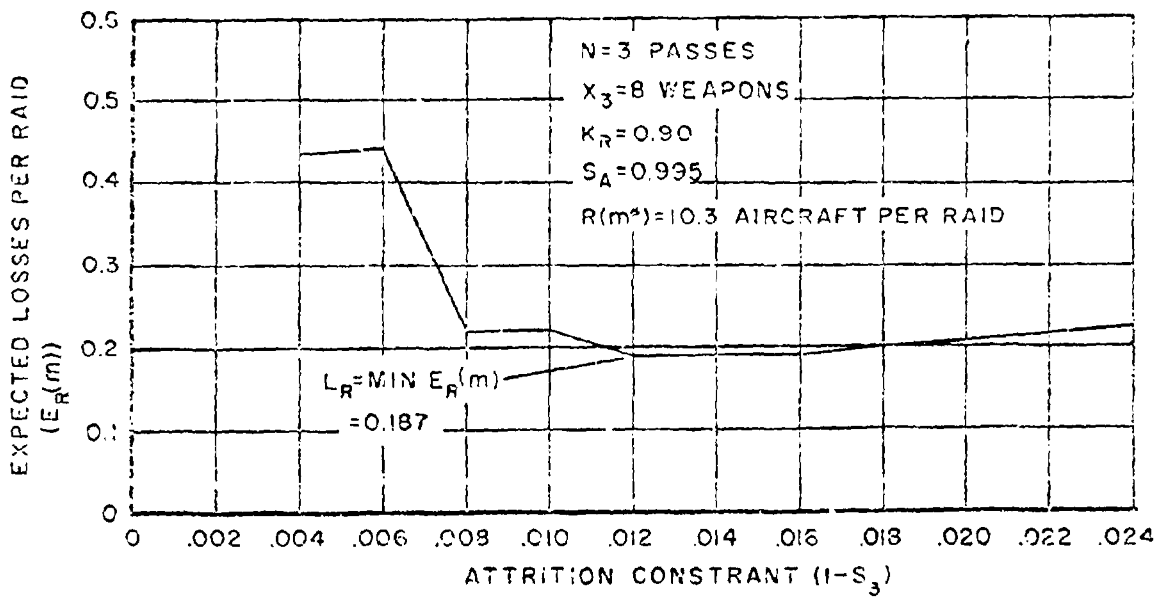


Fig. 22.--Minimizing expected losses in the P_K duel.

To a large extent, the symbols used here are simply the symbols used for the single target raid model with the subscript t added. Thus, $u_{Dt}(m_t)$ and $S_{Dt}(m_t)$, respectively, are the expected hits and the probability of aircraft survival per duel with target t when the aircraft uses attack policy m_t; S_{At} is the probability of surviving the area defenses one way enroute to or returning from target t; R_t is the size of the raid on target t (number of aircraft); $E_{Rt}(m_t)$ is the expected losses per raid on target t when policy m_t is used.

The expected hits on target t when the aircraft use attack policy m_t is

$$C_{Rt}(m_t) = R_t \, S_{At} \, u_{Dt}(m_t) \tag{12}$$

Let $U_t[Z]$ be an arbitrary function

$$0 \leq U_t\,[Z] \leq 1.0 \tag{13}$$

which represents the utility of Z expected hits on target t. Assume the utilities of hits on various targets are additive, and let λ_t be the relative importance of the targets where $\lambda_t \geq 0$ and $\sum_{t=1}^{T} \lambda_t = 1$. Then for a given allocation $(R_1,\text{---},R_T)$ and set of policies $(m_1,\text{---},m_T)$, the utility of all raids is

$$\overline{U}_T = \sum_{t=1}^{T} \lambda_t \, U_t\left[C_{Rt}(m_t)\right] = \sum_{t=1}^{T} \lambda_t \, U_t\left[R_t S_{At} u_{Dt}(m_t)\right] \tag{14}$$

We will suppose that the planner decides to select $(m_1,\text{---},m_T)$ and $(R_1,\text{---},R_T)$ so as to maximize the total utility subject to constraints on the total expected losses, $\overline{E}_{RT}$, and the total number of aircraft, $\overline{R}_T$.

This can be stated symbolically as

$$\text{Maximize} \quad \sum_{t=1}^{T} \lambda_t \; U_t \left[R_t \; S_{At} \; u_{Dt}(m_t) \right] \tag{15}$$

subject to

$$\sum_{t=1}^{T} R_t \leq \overline{R}_t \tag{16}$$

$$\sum_{t=1}^{T} E_{Rt}(m_t) = \sum_{t=1}^{T} R_t \left[1 - S_{At}^2 \; S_{Dt}(m_t) \right] \leq \overline{E}_{RT} \tag{17}$$

$$R_t \geq 0 \qquad ; \qquad 1 \leq t \leq T \tag{18}$$

The solution to this problem might be used to present the operational planner with a plot of $\overline{U}_T$ versus $\overline{E}_{RT}$. Corresponding to each point on this curve is an optimal set of attack policies, $(m_1^*,---,m_T^*)$ and an optimal aircraft allocation $(R_1^*,---,R_T^*)$. It would be up to the operational planner to decide which point on the curve constitutes the most desirable operating point.

Note that there is an alternative to the foregoing procedure. It might also seem reasonable to minimize the total expected losses, $\overline{E}_{RT}$, subject to a constraint on the number of aircraft available, $\overline{R}_T$, and a requirement on the total utility. This procedure would not necessarily utilize $\overline{R}_T$ aircraft if the required total utility is set "low" and there may be no feasible solution if the required total utility is set "high." For these reasons, the previous procedure is selected for development.

The problem stated in equations (15) through (18) can be solved by recursive analysis. It will be treated as a T stage problem with stages $1 \leq t \leq T$. The symbol $\bar{R}_t$ will represent the number of aircraft allotted to targets 1,---,t. The symbol $\bar{E}_{Rt}$ will represent the expected losses in attacking targets 1,---,t. The s vector will be $X_t = (\bar{R}_t, \bar{E}_{Rt})$ and the decision vector will be D_t $_t)$. The function $f_t(X_t)$ will be the maximum utility achievab attacks on targets 1,---,t when X_t is the state of the system at stage t.

The transformation equations are

$$\bar{R}_{t-1} = \bar{R}_t - R_t \tag{19}$$

$$\bar{E}_{R,t-1} = \bar{E}_{Rt} - E_{Rt} \tag{20}$$

Appropriate sets can be defined

$$S_{\bar{R}_t} = \left\{ \bar{R}_t \colon 0 \leq \bar{R}_t \leq \bar{R}_T \right\} \tag{21}$$

$$S_{R_t}(\bar{R}_t) = \left\{ R_t \colon 0 \leq R_t \leq \bar{R}_t \right\} \tag{22}$$

$$S_{\bar{E}_{Rt}} = \left\{ E_{Rt} \colon 0 \leq E_{Rt} \leq \bar{\bar{E}}_{RT} \right\} \tag{23}$$

$$S_{m_t}(\bar{E}_{Rt}, R_t) = \left\{ m_t \colon R_t \left[1 - S_{At}^2 \, S_{Dt}(m_t)\right] \leq \bar{E}_{Rt} \right\} \tag{24}$$

where $\bar{R}_t$, R_t, and m_t are integers, $\bar{E}_{Rt}$ takes only selected discrete values, and $\bar{\bar{E}}_{RT}$ is an arbitrary limiting value. Note that $S_{R_t}(\bar{R}_t)$ depends on $\bar{R}_t$ and $S_{m_t}(\bar{E}_{RT}, R_t)$ depends on $\bar{E}_{Rt}$ and R_t. Equation (24)

says that a mode of attack for target t is chosen from among all modes for which the expected losses in attacking target t is not greater than the expected losses allowed for attacking all of the targets 1,---,t.

The return functions are

$$g_1(X_1,D_1) = \lambda_1 \, U_1 \left[R_1 \, S_{A1} \, u_{D1}(m_1)\right] \qquad ; \qquad t = 1 \tag{25}$$

and

$$g_t \left[X_t, D_t, f_{t-1}(X_{t-1})\right] = \lambda_t \, U_t \left[R_t S_{At} u_{Dt}(m_t)\right] + f_{t-1}(X_{t-1}) \quad ; \quad 2 \leq t \leq T \tag{26}$$

So the functional equations can now be written

$$f_1(X_1) = \max_{\substack{R_1 \in S_{R_1}(\bar{R}_1) \\ m_1 \in S_{m_1}(\bar{E}_{R1}, R_1)}} \left\{ \lambda_1 \, U_1 \left[R_1 \, S_{A1} \, u_{D1}(m_1)\right] \right\} \quad ; \quad t = 1 \tag{27}$$

If we make the assumption that $U_t \left[Z\right]$ is a non decreasing function of Z, then

$$f_1(X_1) = \lambda_1 \max_{m_1 \in S_{m_1}(\bar{E}_{R_1}, \bar{R}_1)} \left\{ U_1 \left[\bar{R}_1 \, S_{A1} \, u_{D1}(m_1)\right] \right\} \quad ; \quad t = 1 \tag{28}$$

and

$$f_t(X_t) = \max_{\substack{R_t \in S_{R_t}(\bar{R}_t) \\ m_t \in S_{m_t}(\bar{E}_{Rt}, R_t)}} \left\{ \lambda_t \, U_t \left[R_t \, S_{At} \, u_{Dt}(m_t)\right] + f_{t-1}(X_{t-1}) \right\} \quad ; \quad 2 \leq t \leq T \tag{29}$$

Equations (28) and (29) can be applied recursively to solve the problem stated in equations (15) through (18).

A multiple target model based on the P_K duel can be developed in a manner similar to the foregoing except that where the foregoing deals with expected hits, we now are interested in probability of kill. Thus, analogous to equation (12), we now express the probability of killing target t with a raid of R_t aircraft each using policy m_t.

$$K_{Rt}(m_t) = \left[1 - S_{At}\, K_{Dt}(m_t)\right]^{R_t} \tag{30}$$

The quantity $K_{Dt}(m_t)$ is the probability of killing target t per duel when policy m_t is used. Where the utility function $U_t\left[C_{Rt}(m_t)\right]$ or $U_t\left[R_t\, S_{At}\, u_{Dt}(m_t)\right]$ appear in equations (14), (15), and (25) through (29), it is replaced by $U_t^K\left[K_{Rt}(m_t)\right]$.

It would be a simple extension of these models to develop a mixed generalized raid model where some of the target attacks would be describable as E_H duels while other target attacks would be describable as P_K duels. It would be necessary only to use the appropriate $U_t\left[\,\cdot\,\right]$ or $U_t^K\left[\,\cdot\,\right]$ for each target in equations (28) and (29).

Special constraints such as minimum required R_t or $S_{Dt}(m_t)$ for various targets can easily be included by simply revising the sets appropriately.

Multiple Aircraft Raid Model Based on the E_D Duel

A reasonable extension of the work that is discussed in this chapter would involve using the E_D duel that was developed in Chapter VI as a basis for a multiple aircraft, single target raid model.

Such a model is conceivable but it would be considerably more complex than the raid models that are discussed in this chapter.

In the E_H duel, the return is in terms of the expected value of the number of hits which implies that the aircraft's utility for the duel is linearly proportional to the number of hits. Accordingly, the optimum policy for a given duel is that policy which maximizes the expected hits subject to suitable constraints. Once having determined the optimum policy for a single duel, we can make the assumption that all aircraft make stochastically independent, statistically identical attacks so that the return from the raid is directly proportioned to the raid size. Thus, we have a simple way of determining the raid size required for a given level of return. The foregoing is the basis of the E_H duel raid model.

In the P_K duel, the return is the probability of at least one hit. The optimum policy for each aircraft in a raid is the policy that maximizes the aircraft's probability of getting at least one hit subject to suitable constraints. Once this policy is determined, we can find a simple relationship between return and raid size, R. If all of the aircraft make stochastically independent, statistically identical attacks, then the probability of the target not being hit at least once is the quantity one minus the probability of at least one hit per aircraft raised to the power R. This provides a simple means of determining the number of aircraft required to achieve a given return and thus we have the basis of the P_K duel raid model.

No such simple relationships as those discussed above seem to exist for the E_D duel. Since the return may be a nonlinear function

of the number of hits, it is not generally appropriate to maximize the expected value of the number of hits for an E_D duel. The optimum policy must reflect the nonlinear character of the utility as a function of the number of hits. Because of the nonlinearity of the utility function, the return associated with hits made by a given aircraft depends on the number of hits made by the other aircraft in the raid.

To see this, suppose two aircraft are to make successive attacks on a target such that the first aircraft completes all of its passes before the second aircraft begins its attack. The character of the utility function that governs the second aircraft's attack depends on the outcome of the first aircraft's attack, i.e., the point at which the second aircraft "enters" the overall utility function depends on the number of hits achieved by the first aircraft. This effect is still present but in a more complex way when the two aircraft alternate in making their passes and also when there are more than two aircraft. Because of this inherent interaction among the E_D duels, there is apparently no rigorous way to develop the optimum policy for a single E_D duel and then use this policy in dealing with multiple aircraft raids.

CHAPTER VIII

USER UNCERTAINTY

General

The discussion in previous chapters and in much of operations research centers is on finding the optimum. Since we have now developed maximizing techniques for at least some air-to-ground attack situations, some degree of satisfaction should have been attained. There is, indeed, some satisfaction in contemplating a maximizing solution but as is often the case, overcoming one obstacle only reveals the greater challenges that lie beyond.

The techniques that have been discussed lead to solutions that are valid for a specific set of input values. Figure 23 summarizes these inputs as they apply to the single target raid models and indicates the outputs that result.

In making actual decisions, there is invariably some degree of uncertainty associated with the values of input parameters. A solution that applies for only one set of input values may be useful as a reference for judging the effect of other input values or might be applied directly if one is willing to ignore uncertainty. In general, however, a systematic, quantitative approach is desirable to make the most rational decision based on the best available information.

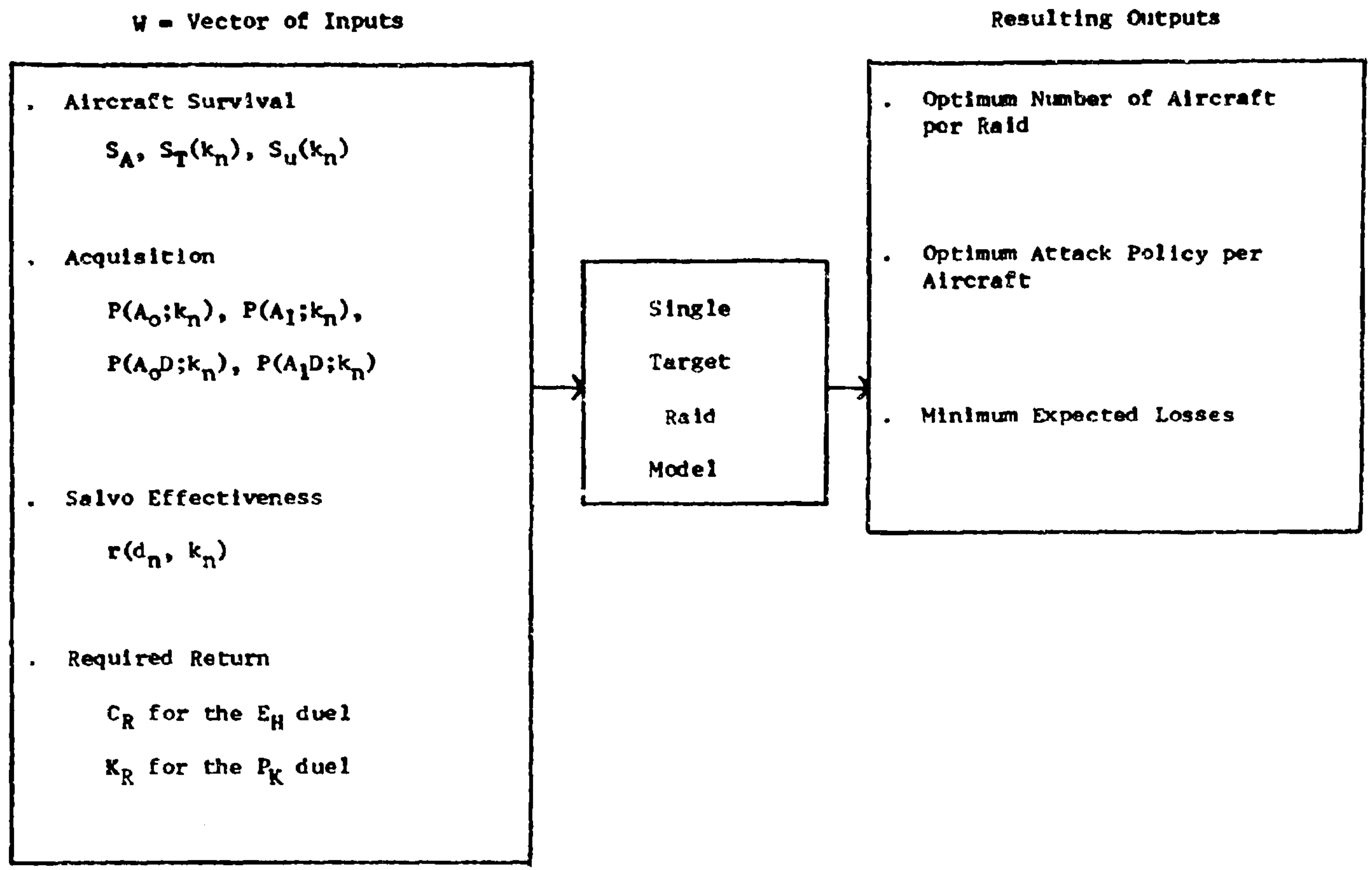


Fig. 23.--Summary of inputs and outputs for the single target raid models.

The Effects of Uncertainty

The approach to be followed in dealing with uncertainty is designed to answer the question: What will happen if the input parameters take values other than the ones upon which the optimization is based?

To answer this question it is first necessary to characterize the available information about the values that the input parameters might take. The specification of intervals of possible parameter variation is one means of reflecting the available information about these parameters. No indication is to be included as to how the input parameter values might vary within their respective intervals. Accordingly, this will be referred to as constrained uncertainty.

Let W be a vector whose components consist of the system inputs. Let the most optimistic values and the most pessimistic values of the input parameters be represented by the vectors W_o and W_p, respectively. Now define α such that

$$W = (1 - \alpha)\, W_p + \alpha\, W_o \qquad ; \qquad 0 \leq \alpha \leq 1 \qquad (1)$$

Thus, W is a convex combination of the vectors W_p and W_o. As the value of α varies from 0 to 1, all of the input parameters vary in unison from their most pessimistic value to their most optimistic value.

It is also possible to apply the foregoing technique to individual input parameters. Suppose the components of W, W_o, W_p are w_j, w_{oj}, w_{pj}, respectively, where $1 \leq j \leq J$. Then a set of α_j values might be defined such that

$$w_j = (1 - \alpha_j)\, w_{pj} + \alpha_j w_{oj}; \quad 0 \leq \alpha_j \leq 1 \; ; \; 1 \leq j \leq J \qquad (2)$$

When $\alpha_1 = \alpha_2 = \cdots = \alpha_J$, then equation (2) is equivalent to equation (1). As the individual α_j's are varied independently over their possible combinations of values, the corresponding input parameters vary over their possible combinations of values.

This method offers the following features which combine to provide a systematic practical way of studying the implications of uncertainty:

1. It tends to place all input variables on a common scale with respect to their range of uncertainty.

2. It polarizes the inputs with respect to their optimistic and pessimistic directions of variation, i.e., for any input parameter, increasing the corresponding α_j results in the parameter taking a more optimistic value.

3. It allows expression of basic inputs in non probabilistic terms.

If the method is to be practical, the results must be comprehendable to the decision maker. This consideration provides a strong argument for emphasizing the use of equation (1), i.e., varying the input parameter values in unison. By doing this, the extremes of system performance are included, some indication of performance at non-extreme input values is obtained, and the results can be expressed in relatively simple form. Accordingly, the balance of the discussion is concerned with the use of equation (1).

A Basis for Tactics Selection

Suppose that having specified w_o and w_p, the tactic that optimizes system performance is chosen based on the input parameter values

$$W' = (1 - \alpha') W_p + \alpha' W_o \tag{3}$$

It would be possible to determine a limiting envelope of system performance by letting α' vary from 0 to 1.0 and optimizing system performance for each different value of α'. As the value of α' varies, the optimizing tactic changes. Unfortunately, only one tactic can be used in a given situation. Of all the tactics that are forthcoming, as α' varies from 0 to 1.0, there is presumably one that is at least as desirable as any other. Corresponding to this tactic is at least one value of α'. Actually, there is generally a range of values of α' corresponding to each tactic because α' is a continuous variable and changes of optimum tactics occur in a discrete manner provided only pure tactics are allowed.[1]

To judge the desirability of the tactic corresponding to a given value of α', we will examine what would happen if the input parameter values that are actually realized differ from the values W' corresponding to α'. The mechanism for accomplishing this is to associate the realized values of input parameters with the control parameter α according to equation (1).

---

[1]Mixed tactics are conceivable here in the same sense that mixed strategies occur in game theory. Only pure tactics are considered.

Let $F(\alpha;\alpha')$ represent a measure of system performance as a function of the realized input values α when tactics optimization is based on the nominal input values corresponding to α'. A plot of $F(\alpha;\alpha')$ versus α for given α' might be considered as a profile of the performance that results from adopting the tactic corresponding to α'.

To illustrate the application of the foregoing technique, suppose $F(\alpha;\alpha')$ represents the cost of doing a fixed job in a given situation. The analysis might result in the performance profiles that appear in Figure 24 for three different values of α'. If $\alpha' = 0.1$, curve a represents the profile of system performance as the input parameters vary in unison through their range. Curves b and c give similar information for $\alpha' = 0.5$ and $\alpha' = 0.9$, respectively. This display presents to the decision maker a highly digested summary of the implications of uncertainty and his options to control the outcome.

We can analyze Figure 24 in terms of the principles of choice under uncertainty (19). The minimax principle leads to selecting the α' that minimizes the cost when $\alpha = 0$. Assuming that all of the α values are equally likely and selecting the α' that gives minimum expected cost is an application of the principle of insufficient reason. Minimizing the maximum difference between the selected curve and the limiting performance envelope, $F(\alpha;\alpha)$ is an application of Savage's principle of minimax regret. Finally, the display itself is in keeping with the Hurwicz pessimism-optimism principle.

In the air-to-ground attack problem and undoubtedly in many other problems, the system performance, $F(\alpha;\alpha')$ is not a scalar quantity but must be considered as a vector. For a given α', both return

and cost vary as α varies. This is unfortunate because it complicates the task of the decision maker in assimilating the results, but it in no way changes the basic idea. Techniques for displaying and interpreting results when $F(\alpha;\alpha')$ is two dimensional will be discussed in relation to the numerical examples.

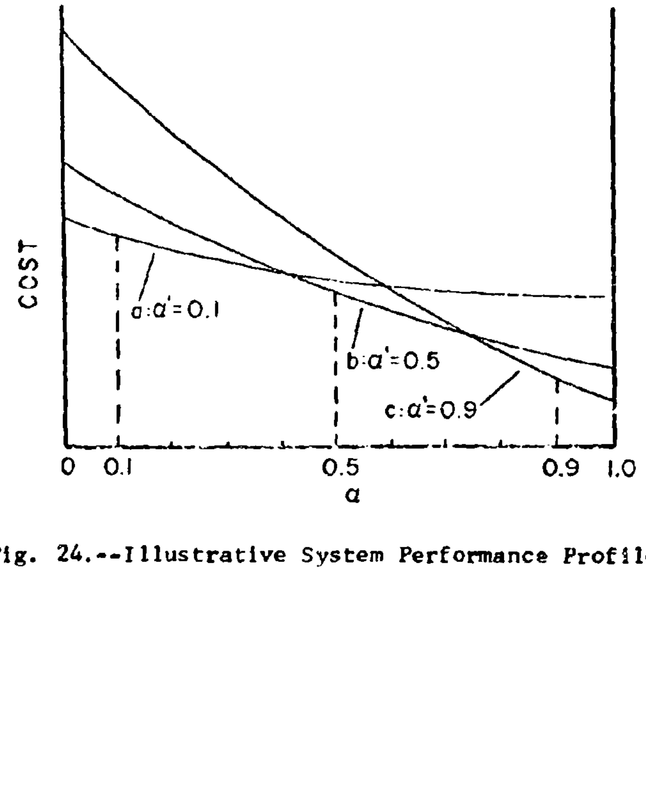


Fig. 24.--Illustrative System Performance Profiles.

Treating Uncertainty in the E_H Duel Raid Model

Consider now the E_H duel of Chapter IV and the corresponding raid model of Chapter VII. The system performance which has been represented by $F(\alpha;\alpha')$ is characterized by two output quantities. Let $L_R(\alpha;\alpha')$ be the expected losses per raid and let $C_R(\alpha;\alpha')$ be the expected hits per raid. These two functions must be evaluated for various values of α'. In this discussion α is associated with realized input values and α' is associated with nominal input values where the tactics optimization is based on the nominal input values.

The number of aircraft per raid is given by equation (VII-3) which becomes

$$R(\alpha') = \frac{C_R}{u_D(\alpha';\alpha')\ S_A(\alpha')} \tag{4}$$

where $u_D(\alpha;\alpha')$ is expected hits per duel as a function of α for given α'; $S_A(\alpha)$ is the probability of surviving area defenses one way as a function of α. Equation (4) simply makes explicit the fact that the raid size is determined entirely from nominal values and is independent of α.

The expected losses realized per raid is given by

$$L_R(\alpha;\alpha') = R(\alpha')\left[1 - S_A^{\,2}(\alpha)\ S_D(\alpha;\alpha')\right] \tag{5}$$

where $S_D(\alpha;\alpha')$ is the probability of the aircraft surviving the duel as a function of α for given α'. The expected hits realized per raid is given by

$$C_R(\alpha;\alpha') = R(\alpha')\ S_A(\alpha)\ u_D(\alpha;\alpha') \tag{6}$$

In evaluating equations (5) and (6) as a function of α, the quantity $S_A(\alpha)$ is an independent input parameter whose value ranges from a pessimistic limit to an optimistic limit in accordance with the value of α. All other input parameter values are also controlled by α and their effect is reflected by the values of $u_D(\alpha;\alpha')$ and $S_D(\alpha;\alpha')$, the expected hits per duel and the aircraft survival probability per duel, respectively.

The functional equations (IV-27) and (IV-29) for the E_H duel can be adapted to evaluate $u_D(\alpha;\alpha')$. Let $d'_{ni}(x_n,s_n;\alpha')$ and $k'_{ni}(x_n,s_n;\alpha')$ be the maximizing attack policy associated with α'. In the following equations, these will be abbreviated d'_n and k'_n respectively. From equation (IV-27), if $n = 1$ and $1 \leq i \leq 3$,

$$f_{1i}(x_1,s_1,\alpha;\alpha') = p_{i3}(k'_1, \alpha)\ S_T(k'_1, \alpha)\ r_H(x_1,k'_1, \alpha) \tag{7}$$

and from equation (IV-29), if $2 \leq n \leq N$ and $1 \leq i \leq 3$,

$$\hat{f}_{ni}(x_n,s_n,\alpha;\alpha') = S_T(k'_n,\alpha)\ S_u(k'_n,\alpha) \sum_{j=1}^{2} p_{ij}(k'_n,\alpha)$$

$$\cdot\ f_{n-1,j}(x_n,s_{n-1}, \alpha;\alpha') + S_T(k'_n, \alpha)\ p_{i3}(k'_n, \alpha) \tag{8}$$

$$\cdot \left[r_H(d'_n, k'_n, \alpha) + S_u(k'_n, \alpha)\ \hat{f}_{n-1,3}(x_n-d'_n,s_{n-1},\alpha;\alpha')\right]$$

where $p_{ij}(k'_n,\alpha)$, $S_T(k'_n, \alpha)$, $r_H(x_n,k'_n, \alpha)$, and $S_u(k'_n, \alpha)$ correspond to $p_{ij}(k_n)$, $S_T(k_n)$, $r_H(x_n,k_n)$, and $S_u(k_n)$ except that they depend on α.[2]

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[2]The "$\wedge$" is used to distinguish the function $\hat{f}_{ni}(x_n,s_n,\alpha;\alpha')$ from the function $f_{ni}(x_n,s_n)$ that has appeared previously. Note that these functions are identical only if $\alpha = \alpha'$.

Since the duel starts with $n = N$, $i = 1$, $x_n = x_N$, and $s_n = s_N$, the expected hits per duel as a function of α for given α' is given by

$$u_D(\alpha;\alpha') = \hat{f}_{N1}(x_N, s_N, \alpha;\alpha') \tag{9}$$

In all calculations, s_{n-1} is related to s_n by the transformation relation (IV-34) which accounts for the discrete nature of the calculations. Note that the survival probabilities to be used in equation (IV-34) are $S_T(k_n', \alpha')$ and $S_u(k_n', \alpha')$. This is because when the pilot carries out a policy, his actions are based on his nominal survival probabilities, not on the actual survival probabilities since he doesn't know the actual values.

To evaluate $S_D(\alpha;\alpha')$ we use the same method that was used in Chapter VII to evaluate the actual survival probability. From equations (VII-1) and (VII-2), if $n = 1$,

$$\phi_{1i}(x_1, s_1, \alpha;\alpha') = S_T(k_1', \alpha)\ S_u(k_1', \alpha) \tag{10}$$

and if $2 \leq n \leq N$

$$\phi_{ni}(x_n, s_n, \alpha;\alpha') = \tag{11}$$

$$\sum_{j=1}^{3} p_{ij}(d_n',\alpha)\ S_T(k_n',\alpha)\ S_u(k_n',\alpha)\ \phi_{n-1,j}(x_{n-1},s_{n-1},\alpha;\alpha')$$

Finally,

$$S_D(\alpha;\alpha') = \phi_{N1}(x_N, s_N, \alpha;\alpha') \tag{12}$$

Treating Uncertainty in the P_K Duel Raid Model

Turning now to the P_K duel of Chapter V, the system performance can be characterized by the expected losses per raid and the probability of killing the target per raid. These quantities will be represented by $L_R(\alpha;\alpha')$ and $K_R(\alpha;\alpha')$, respectively.

The optimal raid size based on the nominal input values is given by the following adaption of equation (V-7)

$$R\ (\alpha') = \frac{\ln(1-K_R)}{\ln\left[1-S_A(\alpha')\ K_D(\alpha';\alpha')\right]} \tag{13}$$

where $K_D(\alpha;\alpha')$ is the probability of kill per duel as a function of α, for given α'. The probability of killing the target per raid as a function of α for given α' is given by

$$K_R(\alpha;\alpha') = 1 - \left[1 - S_A(\alpha)\ K_D(\alpha;\alpha')\right]^{R(\alpha')} \tag{14}$$

and the expected losses per raid as a function of α for given α' is given by

$$L_R(\alpha;\alpha') = \left[1 - S_A{}^2(\alpha)\ S_D(\alpha;\alpha')\right]\ R(\alpha') \tag{15}$$

where $S_D(\alpha;\alpha')$ is the probability of the aircraft surviving the duel as a function of α for given α'.

An adaption of the functional equations for the P_K duel will serve to evaluate $K_D(\alpha;\alpha')$. From equation (V-24), if n = 1 and for $1 \leq i \leq 3$,

$$\hat{f}_{1i}(x_1,\ s_1,\ \alpha;\alpha') = p_{1j}(k_1^i,\ \alpha)\ S_T(k_1^i,\ \alpha)\ r_K(x_1,\ k_1^i,\ \alpha) \tag{16}$$

and from equation (V-26) if $2 \leq n \leq N$ and for all $1 \leq i \leq 3$,

$$\hat{f}_{ni}(x_n, s_n, \alpha;\alpha') = S_T(k_n', \alpha)\, S_u(k_n', \alpha) \sum_{j=1}^{2} p_{ij}(k_n', \alpha)$$

$$\cdot\, \hat{f}_{n-1,j}(x_n, s_{n-1}, \alpha;\alpha') + S_T(k_n', \alpha)\, p_{i3}(k_n', \alpha)$$

$$\cdot\, \left[r_K(d_n', k_n', \alpha) + S_u(k_n', \alpha)\,(1 - r_K(d_n', k_n', \alpha))\right.$$

$$\left.\cdot\, \hat{f}_{n-1,j}(x_n - d_n', s_{n-1}, \alpha;\alpha')\right] \tag{17}$$

Finally,

$$K_D(\alpha;\alpha') = \hat{f}_{N1}(x_N, s_N, \alpha;\alpha') \tag{18}$$

The survival probability $S_D(\alpha;\alpha')$ can be evaluated by using equations (10), (11), and (12). Equation (IV-34) relates s_{n-1} to s_n.

Numerical Example: Tactics Selection Using the E_H Duel Raid Model

The implications of user uncertainty with respect to the E_H duel raid model will now be illustrated by using the example problem that was introduced in Chapter IV and further discussed in Chapter VII. The nominal input values relating to acquisition, aircraft survival, and weapon effectiveness are unchanged from the previous example, but a range of uncertainty will now be associated with each of the input quantities. The optimistic and pessimistic value of each parameter is chosen so that the parameter values used in the example from Chapters IV and VII correspond to $\alpha' = 0.5$.

Figure 25 shows the acquisition probabilities and their range of uncertainty. Figure 26 shows the survival probabilities and their range of uncertainty. Figure 27 shows the salvo effectiveness functions with the range of uncertainty illustrated for mode of attack number four. The optimistic and pessimistic limits for the salvo effectiveness functions were generated by respectively increasing and decreasing the value of the multiplier Ψ by ten per cent of the nominal value.

The modes of attack might be visualized as representing different aircraft attack speeds. Mode one has a higher salvo effectiveness and a lower survival probability which might result from lower speed. Mode four has a lower salvo effectiveness and higher survival probability which might result from higher speed of attack. It appears reasonable to assume that the acquisition probabilities do not depend on the attack speed (13,14,24).

Evaluation of equations (5) and (6) for the values $\alpha' = 0.0$, $\alpha' = 0.5$ and $\alpha' = 1.0$ produces the profiles of system performance that are shown in Figure 28.

With these profiles before him, the operational planner might first observe that of the three values of α', the moderate philosophy represented by $\alpha' = 0.5$ shows lower expected losses, $L(\alpha;\alpha')$, for all values of α than does either of the other values of α'. Looking further, he notes that $\alpha' = 0.5$ has higher expected hits, $C_R(\alpha;\alpha')$, for all values of α than does $\alpha' = 1.0$, so $\alpha' = 1.0$ is clearly dominated by $\alpha' = 0.5$. On the other hand, $\alpha' = 0.0$ gives higher losses but it also gives more hits so the choice between $\alpha' = 0.0$ and $\alpha' = 0.5$ is

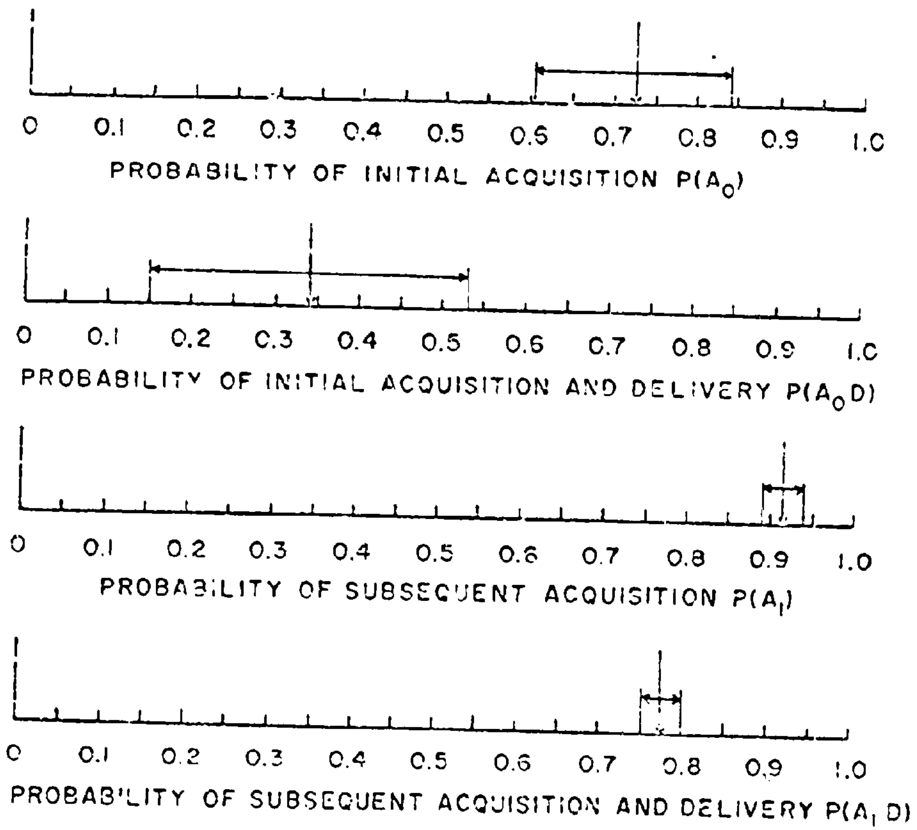


Fig. 25.--Example acquisition probabilities with their ranges of uncertainty.

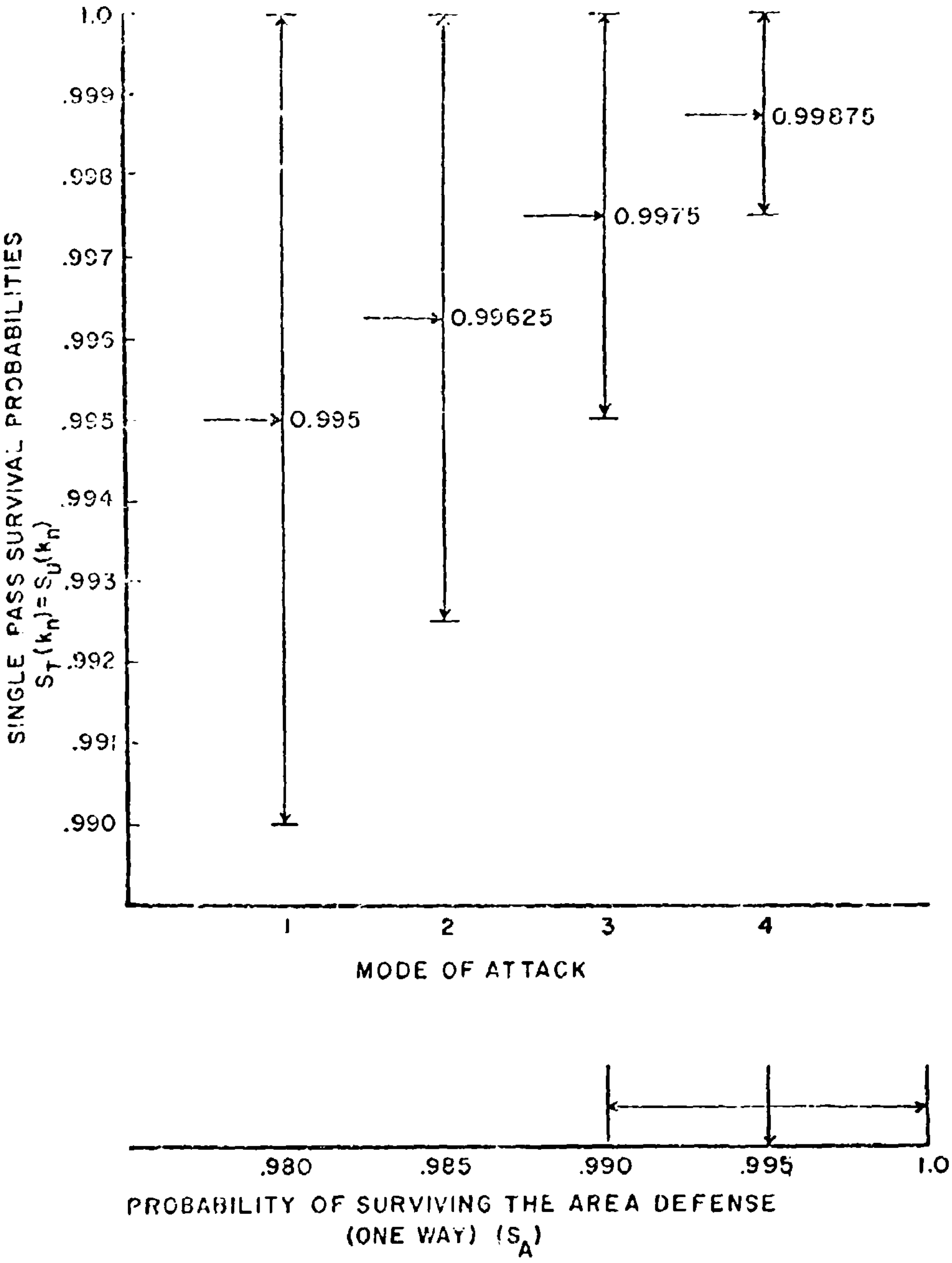


Fig. 26.--Example survival probabilities with their ranges of uncertainty.

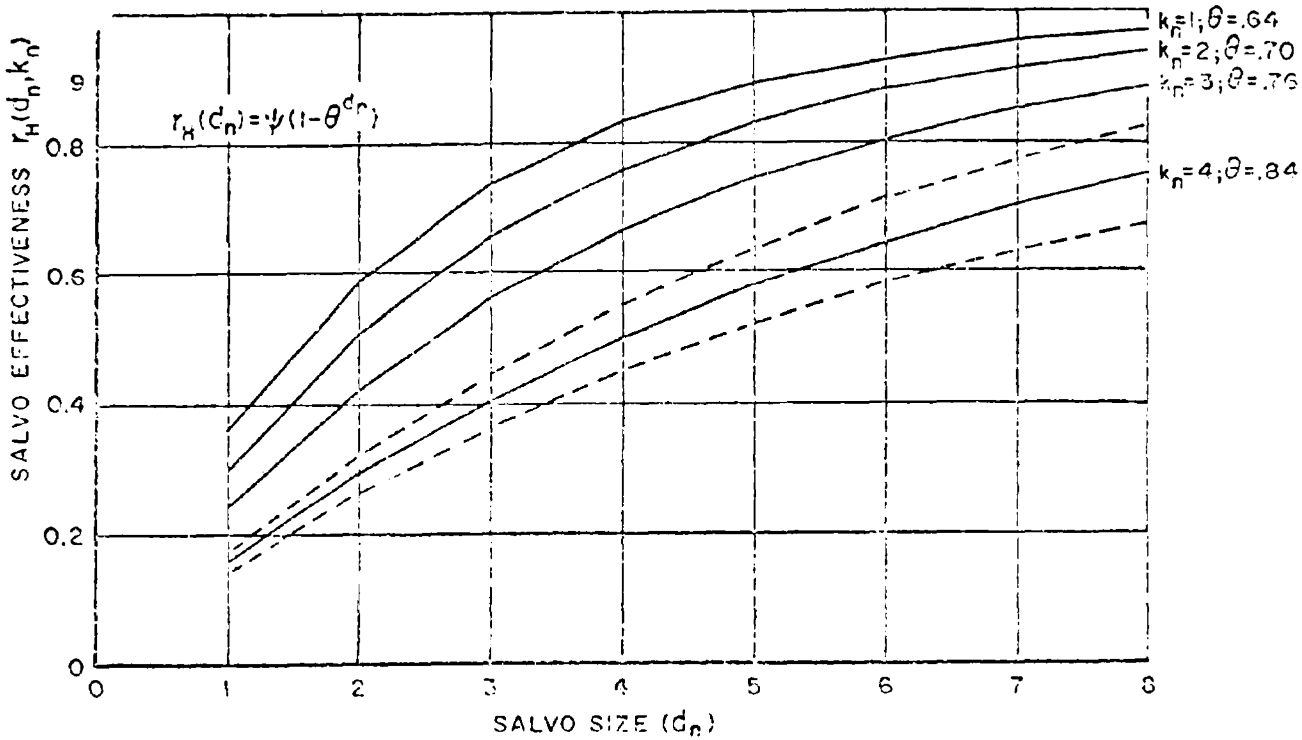
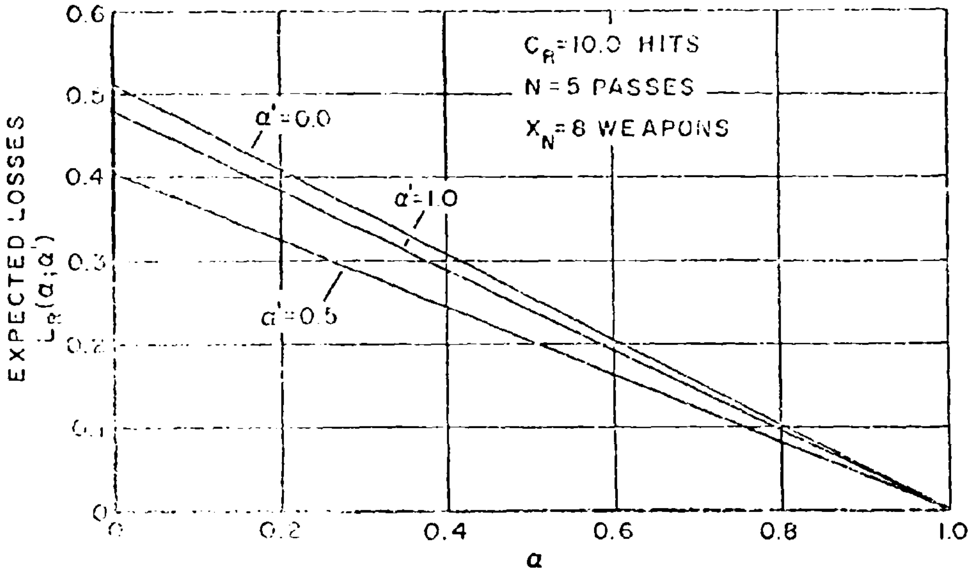


Fig. 27.--Example salvo effectiveness functions.



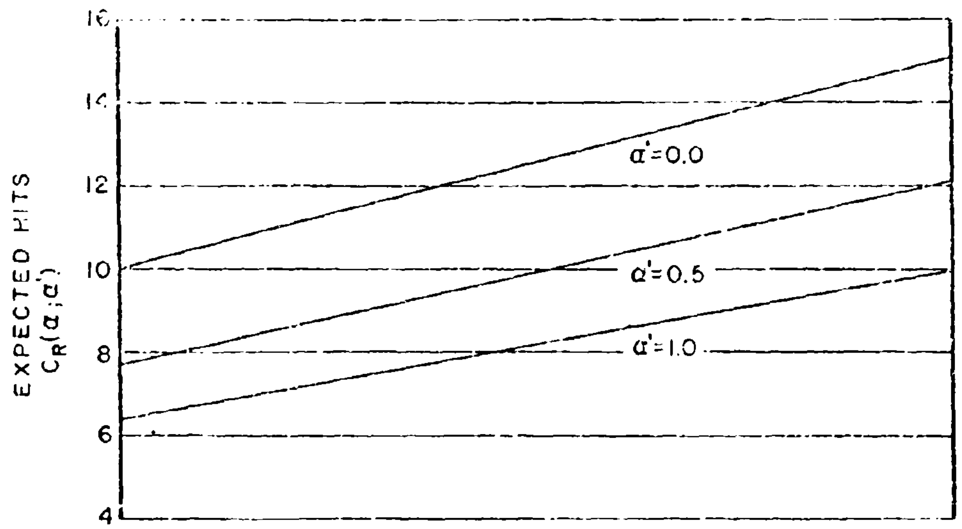


Fig. 28.--Profiles of system performance: E_H duel raid model.

not so clear. Since the goal of the raid is to make ten hits, then the cost of adopting $\alpha' = 0.0$ may not be justified since that philosophy gives ten or more expected hits for all values of α. It might be appropriate to plot profiles for some other values of α' to further illuminate the decision.

It is interesting in general and might be of particular interest to the operational planner to examine the attack policies that are represented by the three values of α'. Figures 29, 30, and 31 are diagrams of the attack policies for $\alpha' = 0.5$, $\alpha' = 0.0$, and $\alpha' = 1.0$, respectively. In this problem the operational planner would be particularly interested in comparing Figures 29 and 30. Note that for $\alpha' = 0.5$ (Figure 29) the minimum number of passes is three, the maximum number of passes is four, and $d_5^* = 3$. For $\alpha' = 0.0$ (Figure 30) the maximum number of passes is four and $d_5^* = 4$. Thus, $\alpha' = 0.0$ leads to a more conservative policy than does $\alpha' = 0.5$ as would be expected. When $\alpha' = 1.0$, on the other hand, the minimum number of passes is five, thus $\alpha' = 1.0$ is the least conservative policy as would be expected. Also, $\alpha' = 1.0$ gives $k_n^* = 1$ for all cases since this is the most effective mode and there is assumed to be no associated attrition penalty.

By reflecting on these results, we can perhaps get some idea of why they occur. Taking $\alpha' = 0.5$ as a base case, lowering α' to 0.0 results in higher d_5^* because of the decreased survival probabilities at $\alpha' = 0.0$. If the probability of survival is less, any weapons not delivered on a given pass are less likely to be delivered, thus the tendency is to deliver the weapons earlier in the duel. Still taking $\alpha' = 0.5$ as a base case, the values of d_n^* are generally lower when

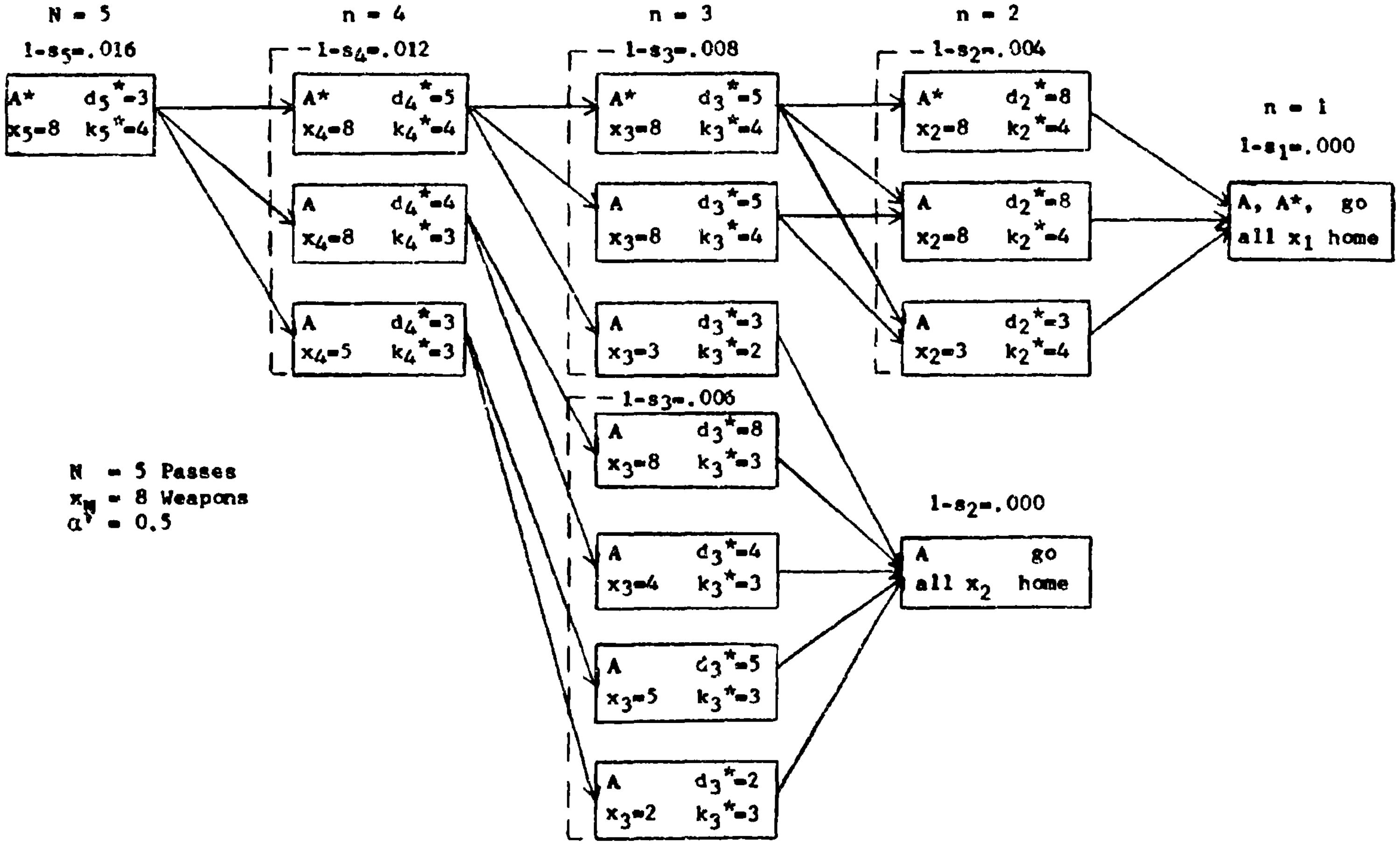


Fig. 29.--Optimum attack policy for the E_H duel raid model: $\alpha' = 0.5$

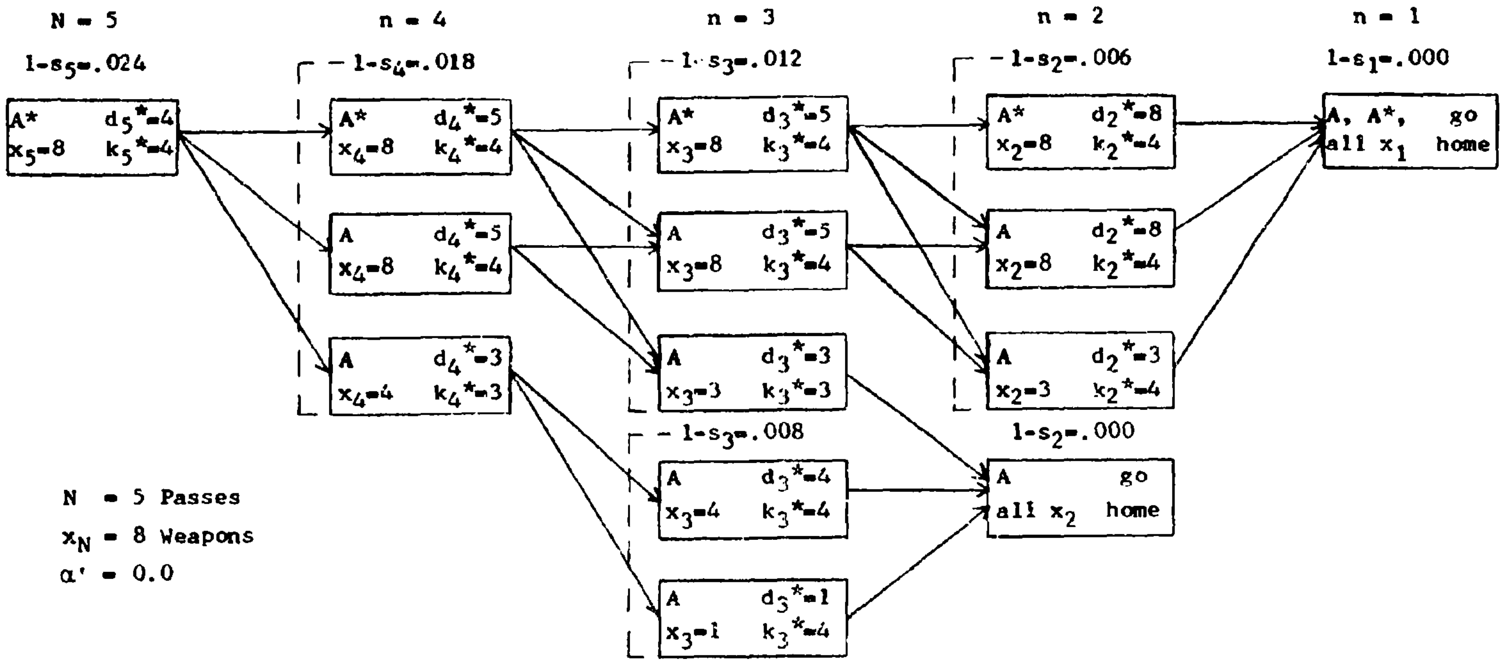


Fig. 30.--Optimum attack policy for the E_H duel raid model: $\alpha' = 0.0$.

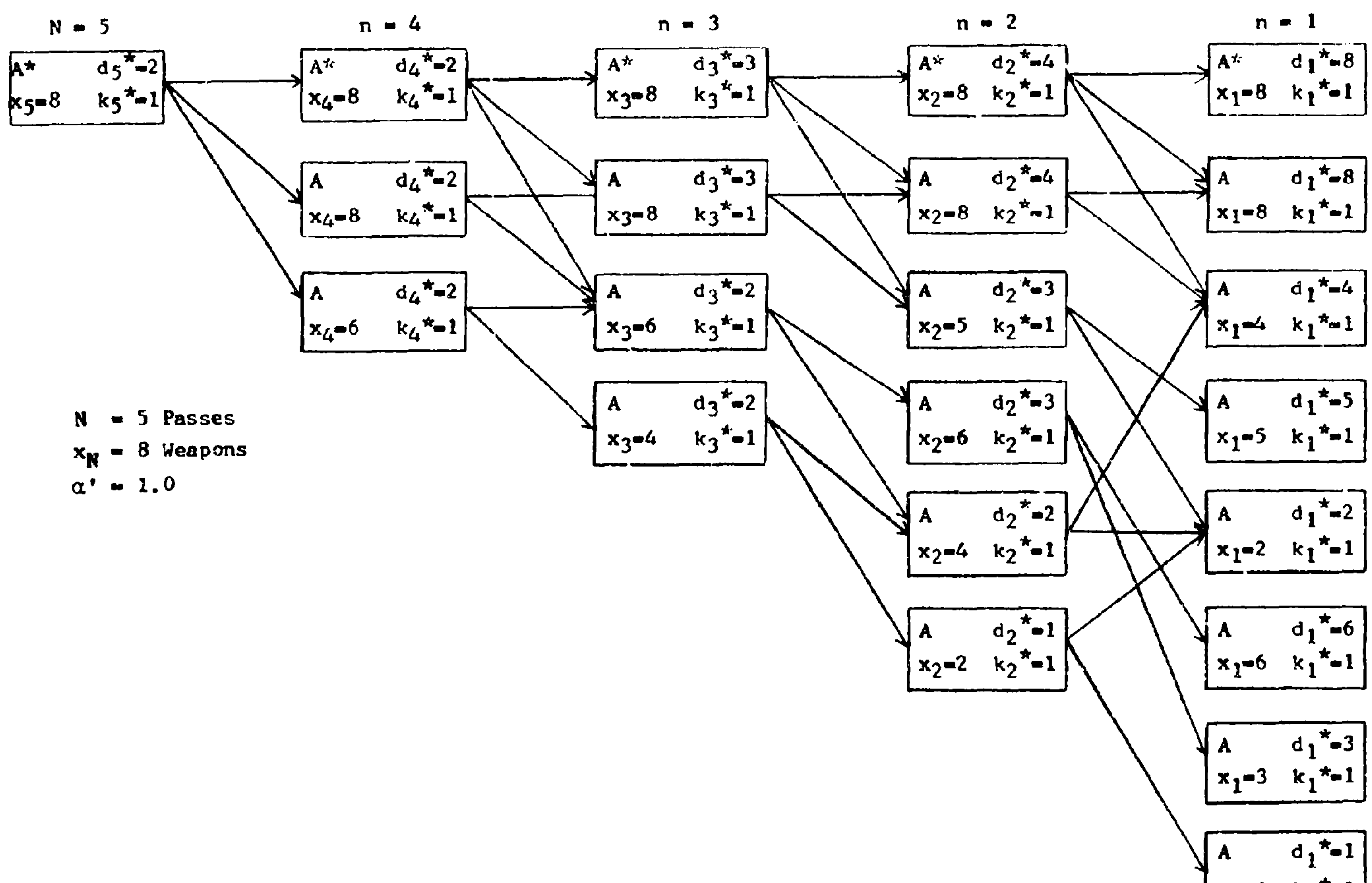


Fig. 31.--Optimum attack policy for the E_H duel raid model: α' = 1.0.

$\alpha' = 1.0$ as shown in Figure 31. This is because the survival probabilities are all equal to 1.0 when $\alpha' = 1.0$ which means that weapons not delivered on the current pass have just as much chance of being delivered on a future pass. There is no discounting since $(S_T S_u) = 1.0$. In all states of Figure 31, the remaining weapons are allocated so they are, as nearly as possible, evenly distributed among the remaining passes. Note in Figure 31 that $k_n = 1$ for all n. This is because mode 1 gives the highest salvo effectiveness and there is no attrition penalty.

Numerical Example: Tactics Selection Using the P_K Duel Raid Model

To illustrate the implications of user uncertainty with respect to the P_K duel raid model, we will use the example that was first introduced in Chapter V and was further discussed in Chapter VII. The acquisition probabilities are given in Figure 25, and the survival probabilities are given in Figure 26. The salvo effectiveness functions are similar to those given in Figure 27 except that the optimistic and pessimistic limits for the multiplier Ψ are now 0.275 and 0.225, respectively. The values of θ are unchanged. The salvo effectiveness function is interpreted here as the probability of kill versus salvo size.

Evaluation of equations (14) and (15) for the values $\alpha' = 0.0$, $\alpha' = 0.5$, and $\alpha' = 1.0$ produces the profiles of system performance that are shown in Figure 32. These curves are very similar to those given in Figure 28 for the E_H duel raid model and the same sort of comments apply.

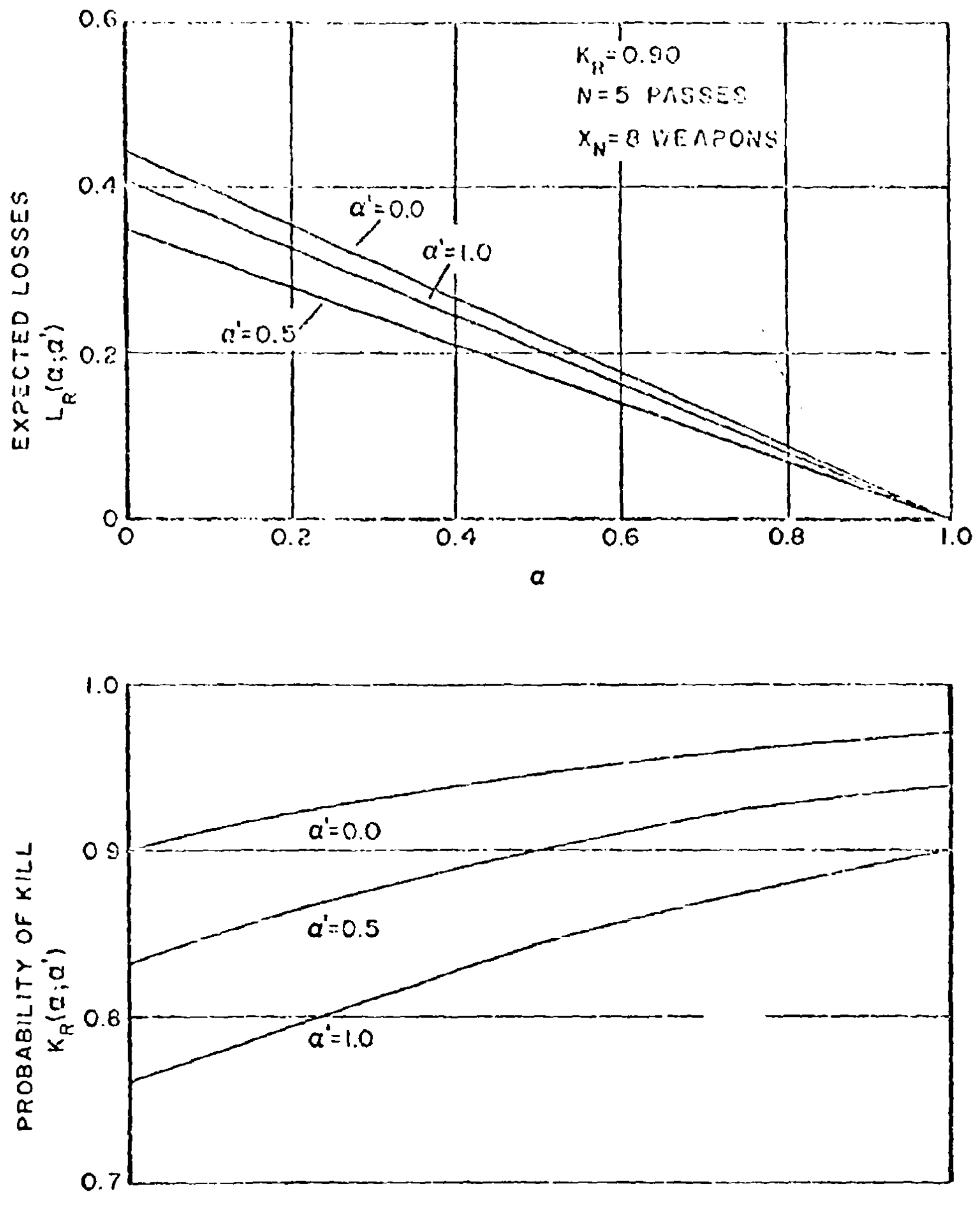


Fig. 32.--Profiles of system performance: P_K duel raid model.

Numerical Example: Revised Survival Probabilities

The system performance profiles in Figures 28 and 32 have the feature that when $\alpha = 1.0$, $L_R(\alpha;\alpha') = 0$. This is because all survival probabilities were set equal to 1.0 at their optimistic limit. It is interesting to see what happens when this is not the case. Suppose that in the E_H duel example, all the inputs are unchanged except that the optimistic limit for S_T and S_u is lowered from 1.0 to 0.9975. This results in the modified probability of survival inputs that are shown in Figure 33. The resulting system performance profiles are shown in Figure 34.

Comparing Figure 34 with Figure 28, the first observation is that losses no longer go to zero when $\alpha = 1.0$. This reflects the reduced optimistic values of S_T and S_u. The values taken at $\alpha = 0.0$ in Figures 34 and 28 are essentially the same but the values taken at $\alpha = 1.0$ have changed from 0.0 in such a way that the expected loss curves now cross. In Figure 28, the tactic for $\alpha' = 1.0$ is clearly dominated by the tactic for $\alpha' = 0.5$, while in Figure 34, there is a part of the range of α values where the expected losses are lower when $\alpha' = 1.0$. Note, however, that for all values of α, the expected hits are much lower for $\alpha' = 1.0$ than for $\alpha' = 0.5$ in both figures. It is not clear which of the three tactics a given decision maker might choose and it is possible that he would like to see results for more values of α'. Nevertheless, the results that have been presented have shed some light on the implications of uncertainty.

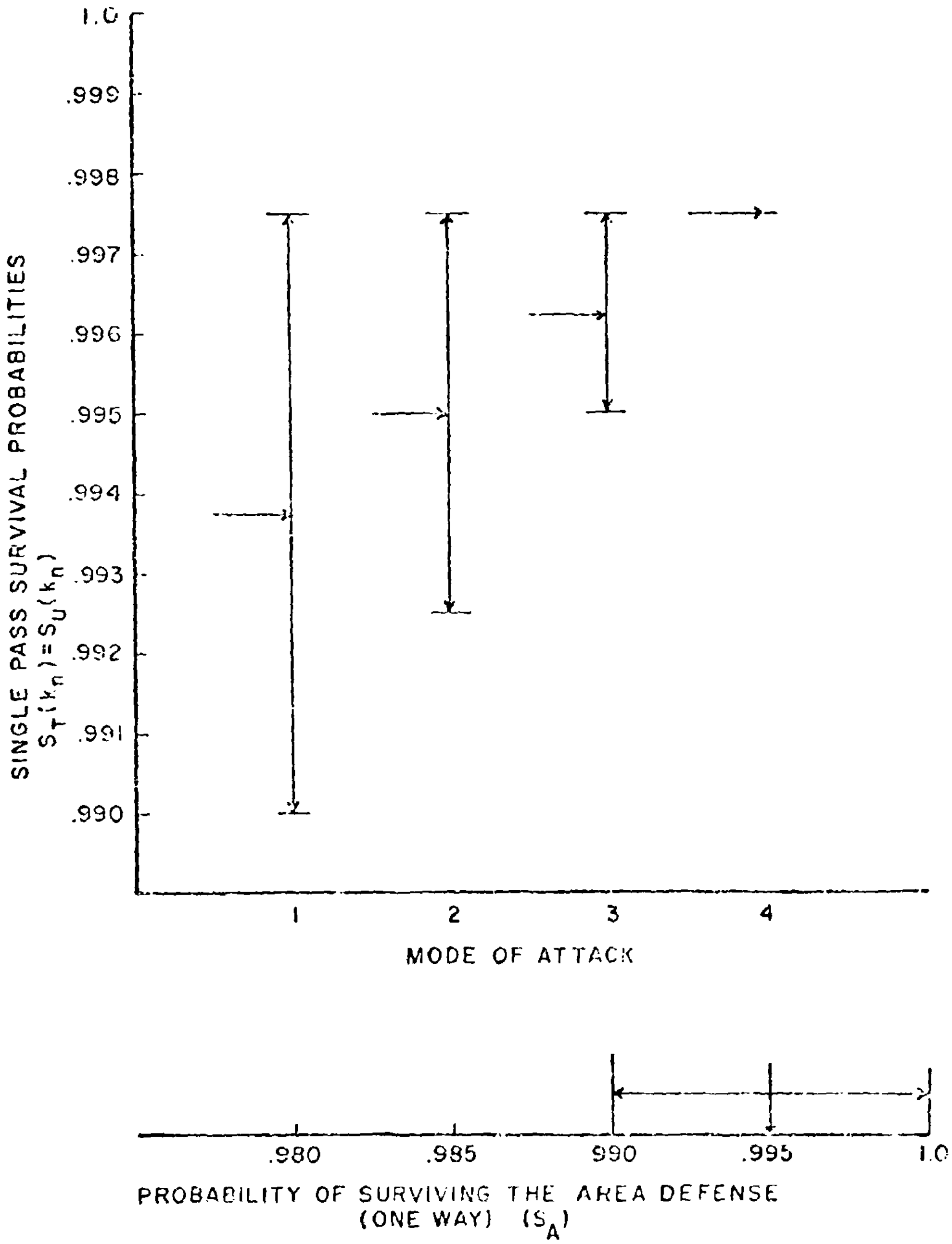


Fig. 33.--Revised survival probabilities.

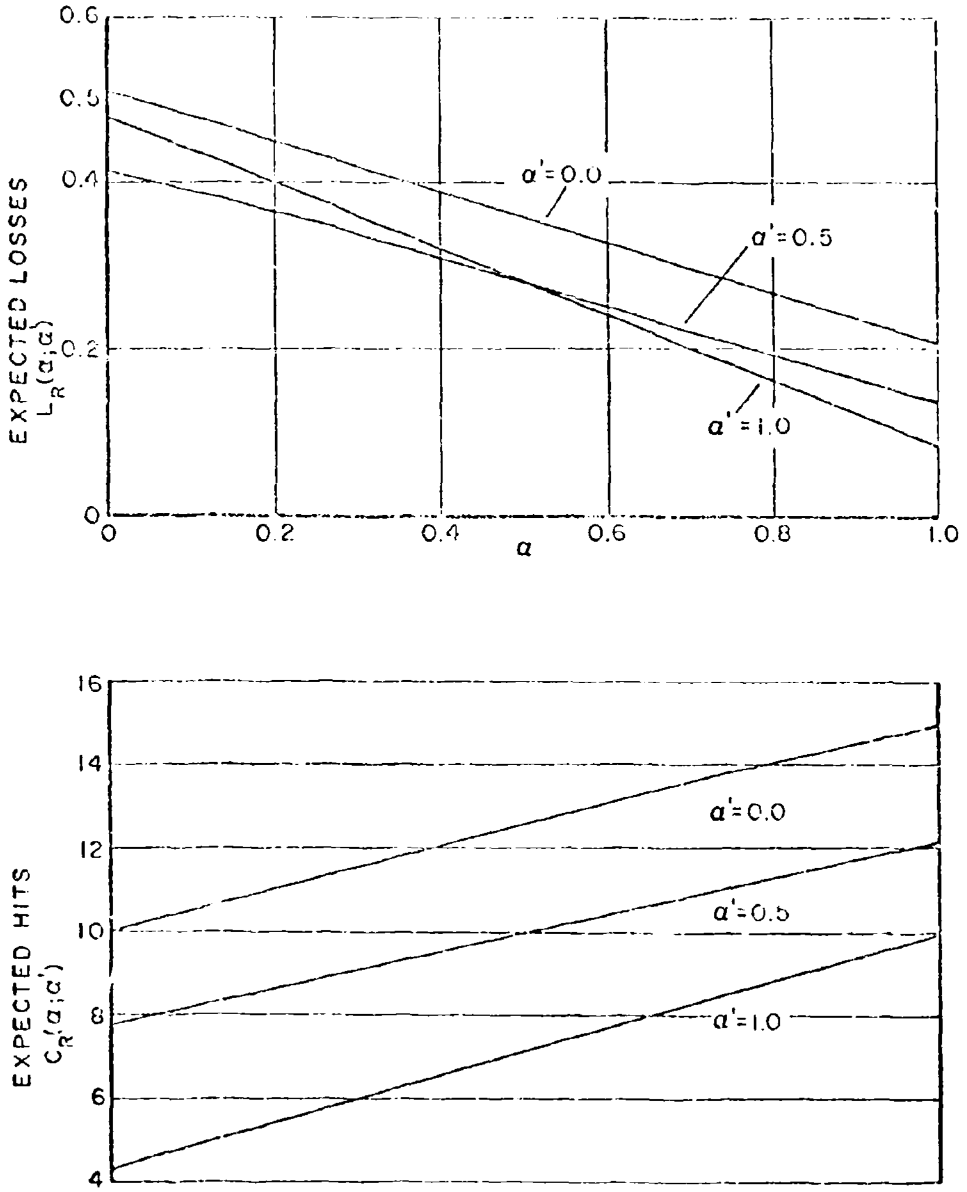


Fig. 34.--Profiles of system performance with revised survival probabilities.

Numerical Example: A Designer's Decision

The previous examples have dealt with the user's decision, i.e., choice of the best tactic for a given system in a given situation. We can begin to consider the designer's decision by way of an example. Suppose the system of the preceding example is taken as a standard system which is to be modified to provide improved performance in that particular situation. Suppose the following alternative designs are available where A represents the standard system while B, C, and D represent equally costly modifications.

A: standard system

B: provide a new sensor: set $P(A_0) = P(A_1) = 1.0$ and $P(A_0D) = P(A_1D)$ = the standard system values for $P(A_1D)$ (see Figure 25)

C: increased weapon load: $x_N = 12$ ($x_N = 8$ in the standard system)

D: improved first pass survival probability: set $S_T(k_n) = S_u(k_n) = 1.0$ on the first pass for all k_n.

When comparing alternatives, we should in principle examine all combinations of alternative and value of α'. The result would be the most desirable combination of alternative and tactic. Since the considerations that are involved are largely subjective, it is difficult to give general rules that would lead to the best decision in a particular situation. Accordingly, it seems reasonable for this example, and perhaps as a starting point for many actual evaluations, to compare alternatives with α' set at a nominal value, say $\alpha' = 0.5$.

Thus, each alternative is evaluated using its own best tactic for the nominal value of input parameters.

Figure 35 shows the profiles of system performance that result for the various alternatives in this example when $\alpha' = 0.5$. These profiles display the effects of uncertainty and the decision maker's options in controlling the outcome. We cannot say how a decision maker would react when confronted by these results, but we can point out some relevant considerations.

First, it is interesting to study the seemingly trivial question of whether or not each alternative actually provides an improvement over the standard system. First, observe that B, C, and D all show lower expected losses than does A for all α. Next, note that C's profile of expected hits is essentially the same as that of A so we might conclude that C's performance is clearly better than A's. If small variation of expected hits is a goal, then B's expected hits profile shows less variation than that of A, therefore, B would be preferred to A. On the other hand, D shows more variation of expected hits than does A and it is conceivable that D would not be preferred to A. This is a very interesting possibility since if uncertainty is entirely suppressed and we compare alternatives on the basis that $\alpha = \alpha' = 0.5$, D would clearly be preferred over A, B, and C. This suggests that if variation of expected hits is of primary importance, then the tactics optimization has been based on an inconsistent criterion. If so, then the problem must be reformulated.

Assume that variation of expected hits is not so important as to force a reformulation of the problem, i.e., alternative D is

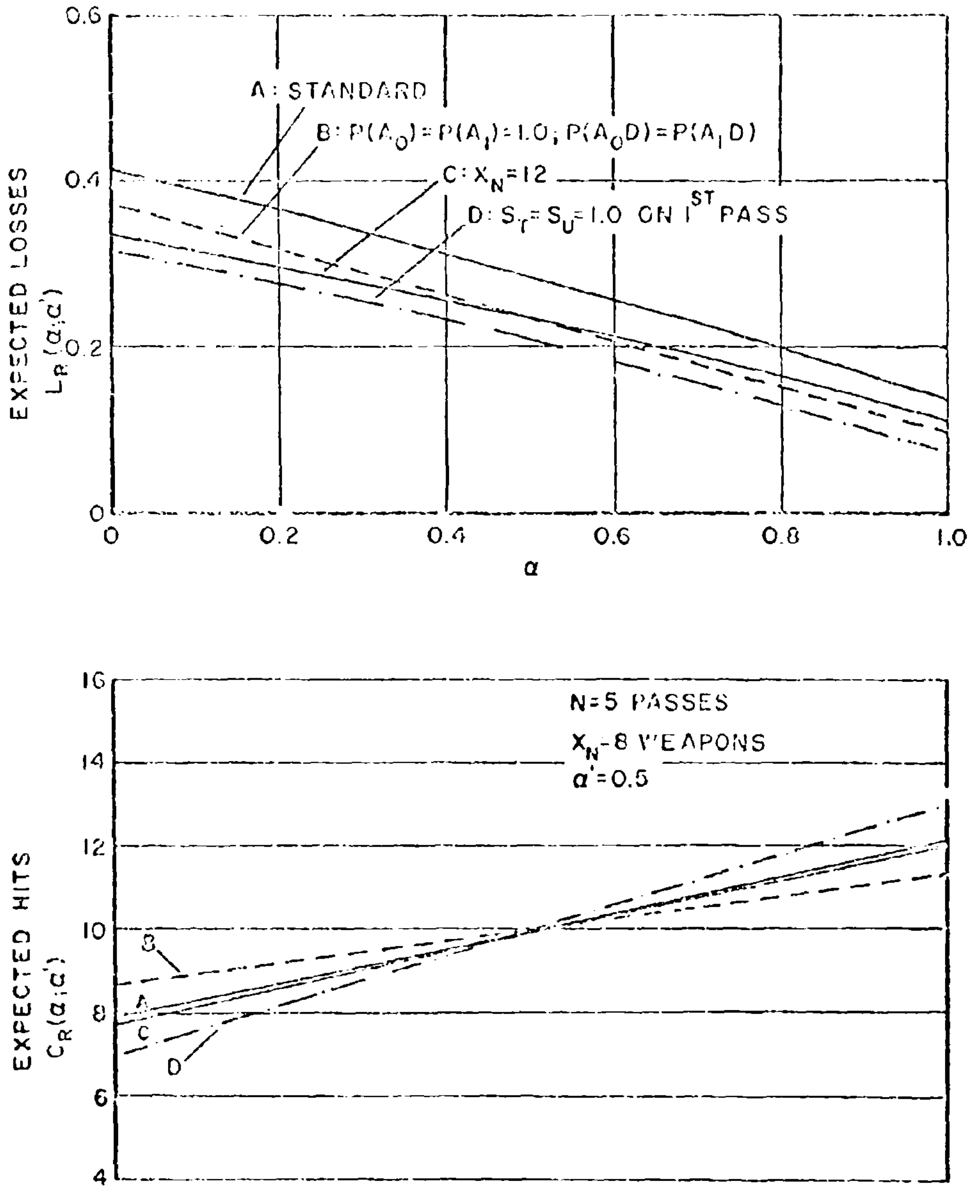


Fig. 35.--Profiles of system performance: alternate system designs.

preferred to alternative A. Even with this assumption, variation of expected hits may be important enough to influence the choice among alternatives B, C, and D. The choice in this example might in fact be viewed as a tradeoff between minimizing expected losses and minimizing the variation of expected hits. To minimize expected losses, alternative D would be selected; alternative B minimizes the variation of expected hits. In this particular case, alternative C might represent a reasonable compromise.

The main point of the foregoing discussion is that if uncertainty had been completely suppressed, alternative D would have been chosen without hesitation. When uncertainty was considered, a new realm of considerations was revealed. Alternative D may no longer be selected. It may even be decided to reformulate the problem. This may or may not be reason to want to quantitatively display the effects of uncertainty, depending on one's point of view. It does, however, illustrate the importance of uncertainty and it shows how the effects of uncertainty can be displayed.

Another interesting point can be made by qualitatively comparing Figure 35 with Figure 34. Alternative A of Figure 35 corresponds to $\alpha' = 0.5$ in Figure 34. As a general observation from comparing these figures, the choice of α' seems to be of comparable importance with the choice of system design. Among the designs and values of α' that were considered, the variation in outcomes caused by changing the design with fixed α' as in Figure 35 seems to be no greater than the variation of outcomes that is caused by changing the value of α' while keeping the same design as in Figure 34.

The importance of tactics can be further emphasized by the following extension of the example. Much current policy for air-to-ground attack establishes a limit of one pass per sortie. This policy would be justified if area defenses were negligible and first pass attrition was small, i.e., the desired level of effectiveness could be achieved by simply increasing the raid size and the losses would be small if only one pass is made per aircraft. Let us see what would happen if a limit of one pass per sortie (N=1) is imposed in the present example where area attrition is not negligible and the enemy has sufficient warning so that first pass attrition is the same as the attrition on later passes. The resulting profiles of system performance are shown in Figure 36 for alternatives A, B, and C. For the standard system, alternative A, the expected losses have roughly doubled and the variation of expected hits has greatly increased.

Suppose as a further extension of the example, system design alternatives B and C are to be compared under the restriction of one pass per sortie. The curves for B and C in Figure 36 are applicable for this comparison. On this basis, B shows considerably lower losses and less variation of expected hits than does C. In Figure 35 where N = 5, the choice between B and C is not so clear.

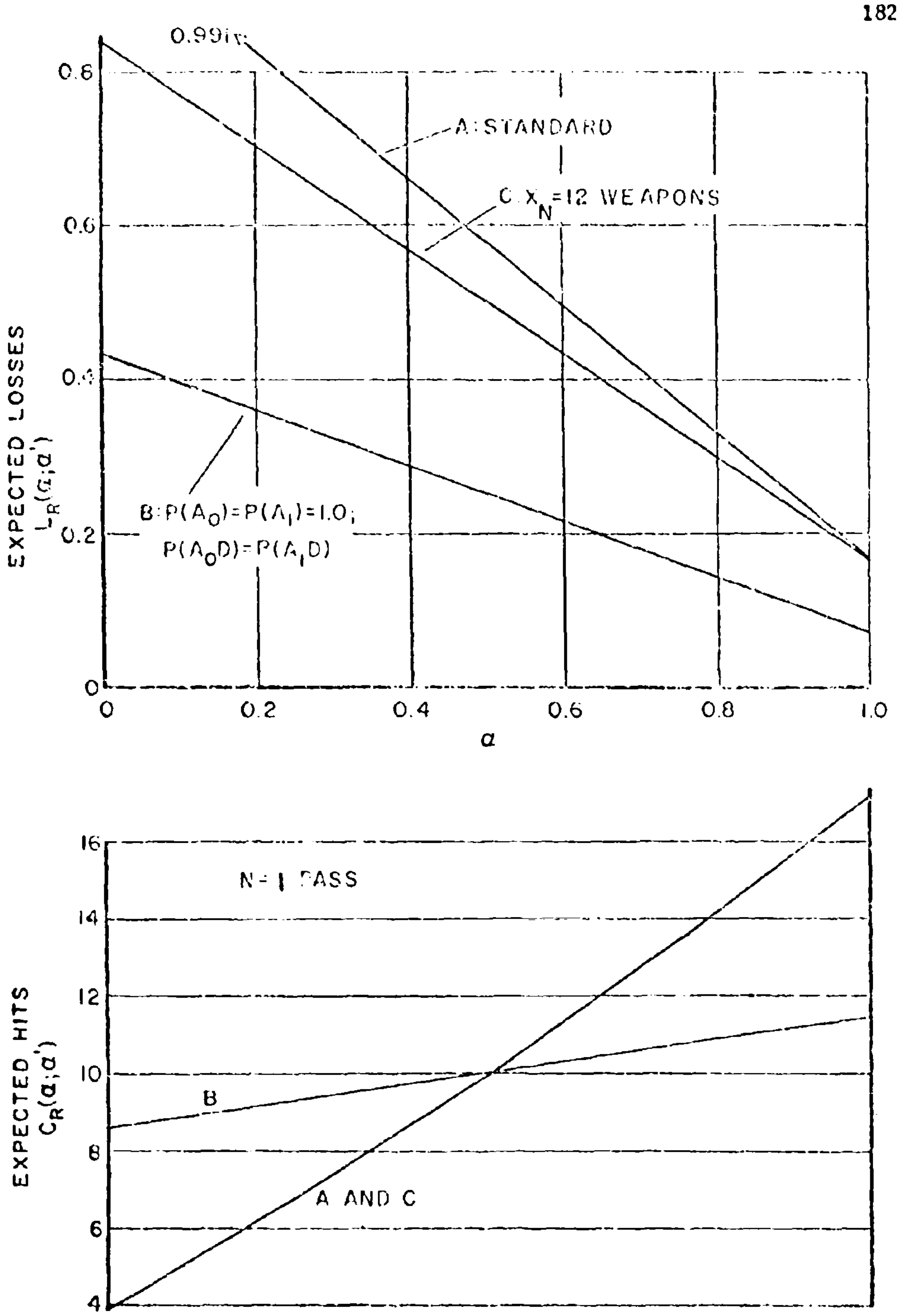


Fig. 36.--Profiles of system performance:

CHAPTER IX

SUMMARY REMARKS

What Has Been Accomplished?

According to the general problem statement given at the beginning of Chapter I, we set out to "provide a systematic method for making the best use of available information" relating to "a rational selection of tactics for air-to-ground attack when faced by uncertainty." The result was to be a "quantitative theoretical structure."

The effort to achieve these goals led first to the discussion of basic recursive analysis techniques. Some general recursive relationships were developed in Chapter II.

It was then shown in subsequent chapters how these recursive relationships can be applied to solve progressively more complex duels. Discussion of these duels started in Chapter III with the simplest case where we sought the allocation of weapons among passes so as to maximize the expected value of the number of hits. Probabilistic target acquisition and multiple modes of attack were added in Chapter IV. In Chapter V the recursive relationships were applied to a duel where the objective was to maximize the probability of at least one hit subject to suitable constraints. In Chapter VI we discussed the more general case where an aspiration level of C hits was established and finally the most general duel was solved in which the

damage level or utility achieved is an arbitrary function of the number of hits.

It was observed that in all of these duels, simply maximizing the return is not sufficient. The cost must be considered. Accordingly, the maximized return for a duel was expressed as a function of the constraining probability of aircraft survival. For convenience, we generally expressed this relationship in terms of attrition rate rather than survival probability and the result was a "return-versus-attrition function" that characterizes the duel. To decide which point on the return-versus-attrition function is the best operating point, multiple aircraft raid models were developed in Chapter VII for both single and multiple target raids.

Finally, in Chapter VIII the effects of user uncertainty were considered and a method was discussed for presenting to the decision maker a display of highly digested information as to the effects of uncertainty and his options to control the outcome.

It is a matter of qualitative judgment as to whether the problem objectives have been achieved, but it certainly seems that progress has been made. A theoretical structure has been developed which provides a systematic way of using available information. Uncertainty has been treated and practical optimizing methods have been developed.

Throughout this discussion, the aim has been to develop methods that can actually be used to analyze real problems. We have attempted not to lose sight of practical application for the sake of mathematical convenience. There are essentially no results that depend on a peculiarity of a functional form. For the most part, the methods used will

accommodate arbitrary functional forms, e.g., salvo effectiveness can take any value within the limits of its definition. The method for dealing with user uncertainty involves only elementary mathematics, but it seems to be practical. It gives useful results without making burdensome demands for input data, i.e., no knowledge of probability theory is required to bound the values of input variables. Considerable thought has been devoted to the selection of criteria. The emphasis has been on finding adequate solution methods based on realistic criteria. This emphasis has led to the consideration of a variety of duels, multiple aircraft raids, multiple target raids, and to user uncertainty, whereas we might have dwelt on finding increasingly elegant solution methods for some problem such as maximizing the expected value of the number of hits in a particular aircraft versus target duel.

In discussing the motivation for this study, we identified a "user's decision problem" and a "designer's decision problem." The results that have been obtained seem to show considerable promise for studying users' decisions. These results should also be useful in studying the designer's decision but we are left with the dilemma that is discussed in the following section.

A Perplexing Problem

A major unsolved problem relates to the effect of user uncertainty on design selection. For a given situation which involves user uncertainty the methods that have been presented in this report can be employed. They serve to determine and portray to the decision maker

the effects of uncertainty and his options to control system performance in that situation. Both tactics selection and design selection can be studied by using these methods.

Unfortunately, in comparing system designs, we must generally consider not just one situation, but an array of situations. In accordance with the introductory discussion, the application of decision theory requires numerical values that represent the utility of each design in each of the situations. The best we have been able to do is to quantitatively display the effects of uncertainty. This provides a basis for the decision maker to choose among alternatives in a given situation but it does not provide the desired utility measure.

How then is the decision maker to make his choice when many situations are involved? One course of action is to limit the number of blocks in the decision matrix and present a complete analysis including the effects of user uncertainty for each block. If the decision matrix is sufficiently small, perhaps the decision maker can comprehend the meaning of such results and make a rational decision accordingly. This approach may at least prevent him from making a completely irrational decision.

An alternative is to entirely suppress the user's uncertainty when making design comparisons (this is what is usually done). The techniques discussed in this report allow us to optimize tactics in each block if user's uncertainty is ignored. This accomplishment is well worthwhile. It reduces our vulnerability to the possibility that tactics are at least as important as system design. Non optimal tactics can be useless or even misleading as a basis for comparing designs.

APPENDIX A

DEVELOPING RECURSIVE RELATIONSHIPS FROM A NON RECURSIVE PROBLEM STATEMENT

In this appendix we will develop a non recursive statement of the most general problem that is treated in Chapter IV, i.e., the E_H duel. The recursive relationships, equations IV-27 and IV-28, which are the principal results of Chapter IV, will then be developed from the non recursive statement of the problem by using a method similar to that of Neuhauser (22). It will also be indicated how the approach that is followed in this appendix can be applied to the most general problem that is treated in Chapter V, i.e., the P_K duel.

In the general E_H duel, we seek to maximize the expected value of the number of hits achieved in a duel that includes probabilistic target acquisition and multiple modes of attack subject to constraints on the number of passes, N, the number of weapons available, x_N, and the probability of the aircraft surviving the duel, s_N.

The notation used in this appendix is the same as that used in the main body of this work with the following modifications. Let j_n denote the Markov state of the system at stage n - 1. Thus the vector $(j_N, j_{N-1}, ---, j_1)$ denotes one of the possible sequences of Markov state transitions and the probability of that sequence occurring is

$$\prod_{n=1}^{N} p_{j_{n+1}j_n}(k_n)$$

where $p_{j_{n+1}j_n}(k_n)$ is the probability that the system will be in state j_n at stage n-1 given that the system is in state j_{n+1} st stage n. This transition probability depends on k_n, the mode of attack at stage n.

Let P_N denote a policy for a duel in which the number of passes is not to exceed N. The policy is a set of instructions that indicates the course of action for the pilot to follow at every stage for every state that the system might be in, i.e., the policy tells the pilot what mode of attack to use and the number of weapons he should seek to deliver on the next pass as a function of the number of passes remaining, the current Markov state, the number of weapons remaining, and the current constraining attrition. We will denote by P'_N that part of the policy that specifies $k_{Nj_{N+1}}(x_N, s_N)$ and $d_{Nj_{N+1}}(x_N, s_N)$ at the first pass. Let P_{N-1} denote that part of the policy that specifies $k_{nj_{n+1}}(x_n, s_n)$ and $d_{nj_{n+1}}(x_n, s_n)$ for all combinations of n, j_{n+1}, x_n, and s_n such that $1 \leq n \leq N - 1$, $1 \leq j_{n+1} \leq I$, $x_n \in S_{x_n}$, and $s_n \in S_{s_n}$. (See Footnotes [1,2]) Let $S_{P'_N}$ denote the set of all possible policies at pass N and let $S_{P_{N-1}}$ denote the set of all possible policies for the remaining N-1 passes.

---

[1]Note that since some of the transition probabilities are zero (see Figure 6), some of the above states may not be reachable, so it is theoretically not necessary for the policy to cover all of them. It is, however, quite complex to determine which of the above states can be reached and which ones cannot and for our purposes, this consideration will be ignored.

[2]As in the main body of this work, the functions $k_{nj_{n+1}}(x_n, d_n)$ and $d_{nj_{n+1}}(x_n, d_n)$ are abbreviated k_n and d_n when they appear in the argument of other functions.

The payoff at stage n depends on whether or not weapons are delivered so we will replace the salvo effectiveness function by $r_{j_n}(d_n,k_n)$, i.e., if $j_n = 1$ or 2, $r_{j_n}(d_n,k_n) = 0$ and if $j_n = 3$, $r_{j_n} = r_H(d_n,k_n)$ where $r_H(d_n,k_n)$ has the same meaning that it did in Chapters III and IV.

For a given sequence of Markov state transitions, $(j_N, ---, j_1)$, and, for a given policy, the expected value of the number of hits is given by the following expression (2). This expression is rationalized by the same type of reasoning that led to equations (III-19) through (III-22) in Chapter III.

$$S_T(k_N)\; r_{j_N}(d_N,k_N) + S_T(k_N)\; S_u(k_N)\; S_T(k_{N-1})\; r_{j_{N-1}}(d_{N-1},k_{N-1})$$

$$+ --- + S_T(k_{N-\ell+1}) \prod_{n=N-\ell+2}^{N} S_T(k_n)\; S_u(k_n)\; r_{j_{N-\ell+1}}\,(d_{N-\ell+1},k_{N-\ell+1})$$

$$+ --- + S_T(k_1) \prod_{n=2}^{N} S_T(k_n)\; S_u(k_n)\; r_{j_1}(d_1,k_1) \qquad (2)$$

The quantity ℓ is a dummy index to be used later. Taking the expectation of the above expression over all sequences of Markov state transitions and maximizing over all possible policies, the objective function becomes expression (3) as shown on the following page.

$$f_{Ni}(X_N) = \max_{\substack{P'_N \in S_{P'_N} \\ P_{N-1} \in S_{P_{N-1}}}} \left\{ \sum_{\text{over } j_N} - - - \sum_{\text{over } j_1} \prod_{n=1}^{N} p_{j_{n+1}j_n}(k_n) \right.$$

$$\left[S_T(k_N)\, r_{j_N}(d_N,k_N) + S_T(k_N)\, S_u(k_N)\, S_T(k_{N-1})\, r_{j_{N-1}}(d_{N-1},k_{N-1}) \right.$$

$$+ - - - + S_T(k_{N-\ell+1}) \prod_{n=N-\ell+2}^{N} S_T(k_n)\, S_u(k_n)\, r_{j_{N-\ell+1}}(d_{N-\ell+1},k_{N-\ell+1})$$

$$\left. \left. + - - - + S_T(k_1) \prod_{n=2}^{N} S_T(k_n)\, S_u(k_n)\, r_{j_1}(d_1,k_1) \right] \right\} \tag{3}$$

where $i = j_{N+1}$ and $X_N = (x_N, s_N)$. The variable i has the same meaning here with respect to stage N that it did in the main body of this work. The variable i indicates the Markov state of the system at stage N.

This expression can be made more compact by using the index ℓ to give

$$f_{Ni}(X_N) = \max_{\substack{P'_N \in S_{P'_N} \\ P_{N-1} \in S_{P_{N-1}}}} \left\{ \sum_{\text{over } j_N} - - - \sum_{\text{over } j_1} \left[\prod_{n=1}^{N} p_{j_{n+1}j_n}(k_n) \right] \right.$$

$$\left. \sum_{\ell=1}^{N} \left[S_T(k_{N-\ell+1}) \prod_{n=N-\ell+2}^{N} S_T(k_n)\, S_u(k_n)\, r_{j_{N-\ell+1}}(d_{N-\ell+1},k_{N-\ell+1}) \right] \right\} \tag{4}$$

where for an arbitrary function g(n) we define

$$\prod_{n=N+1}^{N} g(n) \equiv 1 \tag{5}$$

The above maximization must be carried out subject to two constraints. First, the number of weapons available for n-1 passes, x_{n-1}, is the number of weapons available for n passes, x_n, less the number that is delivered on pass n. Recalling that weapon delivery only occurs at pass n when $j_n = 3$, we have the following relationship for all $1 \leq n \leq N$.

$$
\begin{aligned}
x_{n-1} &= x_n \quad ; \quad j_n = 1, 2 \\
&= x_n - d_n \quad ; \quad j_n = 3
\end{aligned}
\tag{6}
$$

We also require that the probability of the aircraft surviving the duel be at least s_N or symbolically

$$
\prod_{n=1}^{N} S_T(k_n)\, S_u(k_n) \geq s_N \tag{7}
$$

Thus, in the most general problem of Chapter IV, we seek to perform the maximization indicated by equation (4) subject to the constraints indicated by equations (6) and (7).

Isolating the term corresponding to $\ell = 1$ in equation (4) gives

$$
f_{N1}(X_N) = \underset{\substack{P_N' \in S_{P_N'} \\ P_{N-1} \in S_{P_{N-1}}}}{\mathrm{Max}} \Bigg\{ \sum_{\substack{\text{over} \\ j_N}} \cdots \sum_{\substack{\text{over} \\ j_1}} \prod_{n=1}^{N} P_{j_{n+1} j_n}(k_n)
$$

$$
\Bigg[S_T(k_N)\, r_{j_N}(d_N, k_N)
$$

$$
+ \sum_{\ell=2}^{N} S_T(k_{N-\ell+1}) \prod_{n=N-\ell+2}^{N} S_T(k_n)\, S_u(k_n)\, r_{j_{N-\ell+1}}(d_{N-\ell+1}, k_{N-\ell+1}) \Bigg] \Bigg\} \tag{8}
$$

Considering the Markov state transitions, note that the first term of $[\cdot\ \cdot]$ in equation (8) depends only on j_N and the second term depends only on $(j_{N-1},\cdots,j_1)$. Thus, we can write

$$f_{Ni}(X_N) = \underset{\substack{P'_N \in S_{P'_N} \\ P_{N-1} \in S_{P_{N-1}}}}{\mathrm{Max}} \left\{ \sum_{\text{over } j_N} p_{j_{N+1}j_N}(k_N) \left[S_T(k_N)\, r_{j_N}(d_N,k_N) \right.\right.$$

$$+ \sum_{\text{over } j_{N-1}} \cdots \sum_{\text{over } j_1} \prod_{n=1}^{N-1} p_{j_{n+1}j_n}(k_n) \tag{9}$$

$$\left.\left. \sum_{\ell=2}^{N} S_T(k_{N-\ell+1}) \prod_{n=N-\ell+2}^{N} S_T(k_n)\, S_u(k_n)\, r_{j_{N-\ell+1}}(d_{N-\ell+1},k_{N-\ell+1}) \right] \right\}$$

Now by factoring the quantity $S_T(k_N)\, S_u(k_N)$ out of the second term in equation (9) and by replacing the ℓ by $\ell+1$, equation (9) becomes

$$f_{Ni}(X_N) = \underset{\substack{P'_N \in S_{P'_N} \\ P_{N-1} \in S_{P_{N-1}}}}{\mathrm{Max}} \left\{ \sum_{\text{over } j_N} p_{j_{N+1}j_N}(k_N) \left[S_T(k_N)\, r_{j_N}(d_N,k_N) \right.\right.$$

$$+ S_T(k_N)\, S_u(k_N) \left\{ \sum_{\text{over } j_{N-1}} \cdots \sum_{\text{over } j_1} \prod_{n=1}^{N-1} p_{j_{n+1}j_n}(k_n) \sum_{\ell=1}^{N-1} S_T(k_{N-1-\ell+1}) \right.$$

$$\left.\left.\left. \prod_{n=N-1-\ell+2}^{N-1} S_T(k_n)\, S_u(k_n)\, r_{j_{N-1-\ell+1}}(d_{N-1-\ell+1},k_{N-1-\ell+1}) \right\} \right] \right\} \tag{10}$$

Since $S_T(k_N)\, S_u(k_N) \geq 0$, the outside quantity $\{\cdot\cdot\}$ in equation (10) is a monotonically nondecreasing function of inside quantity $\{\cdot\}$,

thus, we have satisfied the monotonicity requirement of Nemhauser (22). Note also that the inside quantity $\left\{ \cdot \right\}$ in equation (10) depends only on the decision variables of P_{N-1}. Thus, in accordance with Nemhauser's proof, equation (10) is equivalent to

$$f_{N1}(X_N) = \underset{P_N' \in S_{P_N'}}{\text{Max}} \left\{ \sum_{\substack{\text{over} \\ j_N}} p_{j_{N+1}j_N}(k_N) \left[s_T(k_N)\, r_{j_N}(d_N, k_N) \right. \right.$$

$$+ s_T(k_N)\, s_u(k_N) \underset{P_{N-1} \in S_{P_{N-1}}}{\text{Max}} \left\{ \sum_{\substack{\text{over} \\ j_{N-1}}} \cdots \sum_{\substack{\text{over} \\ j_1}} \prod_{n=1}^{N-1} p_{j_{n+1}j_n}(k_n) \right.$$

$$\sum_{\ell=1}^{N-1} s_T(k_{N-1-\ell+1}) \prod_{n=N-1-\ell+2}^{N-1} s_T(k_n)\, s_u(k_n)\, r_{j_{N-1-\ell+1}}(d_{N-1-\ell+1},$$

$$\left. \left. \left. k_{N-1-\ell+1}) \right\} \right] \right\} \tag{11}$$

By examining the constraints, equations (6) and (7), it is clear that for given $k_{Nj_{N+1}}(x_N, d_N)$ and $d_{Nj_{N+1}}(x_N, d_N)$, the remaining decision variables must satisfy the constraints

$$\begin{aligned} x_{N-1} &= x_N \qquad ; \quad j_N = 1,2 \\ &= x_N - d_N \quad ; \quad j_N = 3 \end{aligned} \tag{12}$$

and

$$\prod_{n=1}^{N-1} s_T(k_n)\, s_u(k_n) \geq \frac{s_N}{s_T(k_N)\, s_u(k_N)} \equiv s_{N-1} \tag{13}$$

so the second maximization in equation (11) is subject to these constraints.

By replacing N by N-1 in equation (4), we recognize that the quantity $\underset{P_{N-1} \in S_{P_{N-1}}}{\text{Max}} \left\{ \cdot \right\}$ that appears in equation (11) is the same as $f_{N-1,j_N}(X_{N-1})$ so equation (11) can be written as:

$$f_{Ni}(X_N) = \underset{P_N' \in S_{P_N'}}{\text{Max}} \left\{ \sum_{j_N} p_{j_{N+1}j_N}(k\ ) \left[S_T(k_N)\ r_{j_N}(d_N,k_N) \right.\right.$$

$$\left.\left. + S_T(k_N)\ S_u(k_N)\ f_{N-1,j_N}(X_{N-1}) \right]\right\} \tag{14}$$

where $X_{N-1} = (x_{N-1}, s_{N-1})$.

To see that equation (14) is the same as equation (IV-28) for n=N, we use the fact that $r_1(d_N,k_N) = r_2(d_N,k_N) = 0$, while $r_3(d_N,k_N) = r_H(d_N,k_N)$. Substituting these into equation (14) and using equations (12) and (13) and noting the definitions of P_N' and $S_{P_N'}$ gives

$$f_{Ni}(x_N,s_N) = \underset{\substack{k_N \in S_{k_N}(s_N) \\ d_N \in S_{d_N}(x_N)}}{\text{Max}} \left\{ \sum_{j_N=1}^{2} p_{ij_N}(k_N)\ S_T(k_N)\ S_u(k_N) f_{N-1,j_N}(x_N, \right.$$

$$\left. \frac{s_N}{S_T(k_N)S_u(k_N)}) + p_{i3}(k_n) \left[S_T(k_N)\ r_H(d_N,k_N) \right.\right.$$

$$\left.\left. + S_T(k_N)\ S_u(k_N)\ f_{N-1,3}(x_N - d_N, \frac{s_N}{S_T(k_N)S_u(k_N)}) \right]\right\} \tag{15}$$

The trivial generalization of replacing N by n shows that equation (15) holds for all $2 \leq n \leq N$. Furthermore, if for all $1 \leq j_1 \leq 3$,

$$f_{oj_1}(X_o) \equiv 0 \tag{16}$$

Then equation (15) becomes

$$f_{11}(x_1,s_1) = \max_{\substack{k_1 \in S_{k_1}(s_1) \\ d_1 \in S_{d_1}(x_1)}} \left\{ p_{13}(k_1)\ S_T(k_1)\ r_H(d_1,k_1) \right\} \tag{17}$$

which is the same as equation (IV-27).

Now consider how the approach that has been followed in this appendix can be applied to the most general problem that is treated in Chapter V, i.e., the P_K duel. The major change from the previous discussion in this appendix is that we must replace expression (2) with the following expression (18) which represents the probability of at least one hit for a given sequence of Markov state transitions, $(j_N, ---, j_1)$ and for a given policy. Note that for the P_K duel, if $j_N = 3$, then $r_3(d_n,k_n) = r_K(d_n,k_n)$. This expression can be rationalized by using the same type of reasoning that led to equations (5) through (8) of Chapter V.

$$S_T(k_N) r_{j_N}(d_N,k_N) + S_T(k_N) S_u(k_N) S_T(k_{N-1}) \left[1-r_{j_N}(d_N,k_N)\right] r_{j_{N-1}}(d_{N-1},k_{N-1})$$

$$+ S_T(k_N) S_u(k_N) S_T(k_{N-1}) S_u(k_{N-1}) S_T(k_{N-2}) \left[1-r_{j_N}(d_N,k_N)\right]$$

$$\left[1-r_{j_{N-1}}(d_{N-1},k_{N-1})\right] r_{j_{N-2}}(d_{N-2},k_{N-2})$$

$$+$$
$$\vdots$$

$$+ S_T(k_{N-\ell+1}) \prod_{n=N-\ell+2}^{N} S_T(k_n) S_u(k_n) \left[1-r_{j_n}(d_n,k_n)\right] r_{j_{N-\ell+1}}(d_{N-\ell+1},k_{N-\ell+1})$$

$$+$$
$$\vdots$$

$$+ S_T(k_1) \prod_{n=2}^{N} S_T(k_n) S_u(k_n) \left[1-r_{j_n}(d_n,k_n)\right] r_{j_1}(d_1,k_1) \tag{18}$$

The details of the development will not be given but the argument is identical to that given for the E_H duel. The changes that must be made to apply the argument of this appendix to the P_K duel follow in a straightforward manner from the use of expression (18) in place of expression (2).

APPENDIX B

SOME PROOFS RELATED TO THE MATERIAL IN CHAPTER II

This appendix contains a proof of the comment that immediately follows the statement of the equivalence condition for the deterministic decision process in Chapter II, page 31. This appendix also contains a proof of the comment that immediately follows the statement of the monotonicity and equivalence conditions for the Markovian decision process in Chapter II, page 36.

First, relating to the deterministic decision process, we will prove that "it follows from the definitions of monotonicity and $f_{n-1}(X_{n-1})$ that the function $g\left[X_n, D_n, f_{n-1}(X_{n-1})\right]$ represents the maximum return that is obtainable from the n stage system for given X_n and D_n."

Let P_n denote a sequence of decisions, $(D_n, --- D_1)$, or "policy" for stages n,---,1. Let $S_{P_n}(X_n)$ denote the set of all feasible policies for stages n,---,1 where $S_{P_n}(X_n)$ depends on X_n. Let $f'_n(P_n)$ denote the n stage return that is realizable by using policy P_n and let $P_n^*(X_n)$ be the optimum n stage policy as a function of X_n (to be abbreviated P_n^*) so that considering the definition of $f_n(X_n)$,

$$f'_{n-1}(P_{n-1}) \leq f'_{n-1}(P_{n-1}^*) = f_{n-1}(X_{n-1}) \qquad (1)$$

for all $P_{n-1} \in S_{P_{n-1}}(X_{n-1})$.

Now, considering the definition of monotonicity, it follows that

$$g_n[X_n, D_n, f'_{n-1}(P_{n-1})] \leq g_n[X_n, D_n, f_{n-1}(X_{n-1})] \qquad (2)$$

for all $P_{n-1} \in S_{P_{n-1}}(X_{n-1})$ and for every $X_n \in S_{X_n}$ and $D_n \in S_{D_n}(X_n)$. This completes the proof.

Next, relating to the Markovian decision process, we will prove that given monotonicity and equivalence, "the function $g_{nij}[X'_n, D_n, f_{n-1,j}(X_{n-1})]$ represents the maximum expected value of return that is obtainable from the n stage system for given values of X'_n, D_n, i, and j."

Let P_n denote a policy for stages n,---,1. Since the sequence states that the system will occupy at stages n-1,---,1 is not known, the policy must completely define the value that the decision vector is to take as a function of the state of the system at all stages, i.e., P_n must define $(D_n(X_n), D_{n-1}(X_{n-1}),---, D_1(X_1))$ for all feasible sequences $(X_n,---,X_1)$ where X_n is the state vector which includes the Markov state. Let $S_{P_n}(X_n)$ denote the set of all feasible policies for stages n,---,1 where $S_{P_n}(X_n)$ depends on X_n. Let $f'_n(P_n, X_n)$ denote the expected value of the n stage return that is realizable if the system is in state X_n at stage n and policy P_n is used. Let $P_n^*(X_n)$ be the optimum n stage policy as a function of X_n (to be abbreviated P_n^*), so considering the definition of $f_{n-1}(X_{n-1})$,

$$f'_{n-1}(P_{n-1}, X_{n-1}) \leq f'_{n-1}(P^*_{n-1}, X_{n-1}) = f_{n-1}(X_{n-1})$$

for all $P_{n-1} \in S_{P_{n-1}}(X_{n-1})$, $X_{n-1} \in S_{X_{n-1}}$.

Now if j designates the Markov state of the system at stage n-1, the foregoing can be written

$$f'_{n-1,j}(P_{n-1}, X'_{n-1}) \leq f'_{n-1,j}(P^*_{n-1}, X'_{n-1}) = f_{n-1,j}(X'_{n-1}) \quad (3)$$

for all $P_{n-1} \in S_{P_{n-1}}(X'_{n-1})$, $X'_{n-1} \in S_{X'_{n-1}}$, $1 \leq j \leq I$.

It follows from the above and from the definition of monotonicity that

$$g_{nij}\left[X'_n, D_n, f'_{n-1,j}(P_{n-1}, X'_{n-1})\right] \leq g_{nij}\left[X'_n, D_n, f_{n-1,j}(X'_{n-1})\right] \quad (4)$$

for all i, j, X'_n, D_n, P_{n-1} in their respective sets. From the foregoing and the equivalence property, it follows that $g_{nij}\left[X'_n, D_n, f_{n-1,j}(X'_{n-1})\right]$ is the maximum expected value of the n stage return for given X'_n, D_n, i, j. This completes the proof.

APPENDIX C

ANOTHER SOLUTION METHOD FOR MARKOVIAN DECISION PROCESSES

It may have occurred to the reader that the general E_H duel, equations (IV-27) and (IV-29) and the general P_K duel, equations (V-24) and (V-26) can also in principle be solved by Howard's value iteration method (17). It is interesting to see what is involved if that method is applied directly to the examples that have been discussed herein. Howard's notation will be used in this discussion.

To apply value iteration to a discounted Markovian decision problem, the following recursive relation is used (17, p. 80).

$$v_i(n+1) = \max_k \left[q_i^k + \beta \sum_{j=1}^{N} p_{ij}^k \, v_j(n) \right] \qquad (1)$$

where

$q_i^k = \sum_{j=1}^{N} p_{ij}^k \; r_{ij}^k$

n = stage index

i = Markov state; $i = 1, \cdots, N$

k = index on decision alternatives

β = discount factor; analogous to aircraft survival probability

p_{ij}^k = i to j transition probability under decision k

r_{ij}^k = reward associated with the i to j transition under decision k.

To solve the E_H duel example of Chapter IV by use of equation (1), the value of the Markov state variable i must completely characterize the state of the system. In the example of interest, the state of the system is characterized by

a. x_n = The number of bombs remaining: 8 levels.

b. s_n = The constraining survival probability: 12 levels.

c. The acquisition status: 2 levels.[1]

Since these variable values can occur in all combinations, the number of levels required for the Markov state variable is $N = 8 \times 12 \times 2 = 192$. Thus, to solve the example problem by value iteration, each of the transition and reward matrices has $192 \times 192 = 36{,}864$ elements that must somehow be evaluated and accounted for in the calculations.

Solving the Chapter V example of the P_K duel involves even larger size matrices. The state of the system at any stage is characterized by all of the previous factors in addition to which the status of the target must be specified, i.e., it is either dead or alive. This is required because the pilot does not know when target kill is achieved and therefore the duel may continue after the target has been killed but with a different reward per stage. The result is that for the P_K duel example, $N = 8 \times 12 \times 2 \times 2 = 384$ levels are required for the Markov state variable. The transition and reward matrices each have $384 \times 384 = 147{,}456$ elements to somehow account for.

---

[1]Note that for this formulation the acquisition status can be characterized by two levels, i.e., the target has either been acquired or it has not been acquired. In the example of Chapter IV, the acquisition status required three characterizing levels because of the relationship between acquisition status and weapon delivery.

In solving the preceding examples by the methods presented herein, the Markov state variable has three levels and the matrix of transition probabilities has nine elements. The extent of the computation seems to be roughly comparable otherwise. In effect, our method is equivalent to equation (1). The two methods perform the same operations and arrive at the same result, but the former is considerably more efficient and easier to apply to the problems that are of interest here.

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13. ABSTRACT

This work applies dynamic programming and some notions from decision theory to the problem of making a rational selection of tactics for air-to-ground attack. A single aircraft attack on a target is treated as a multistage decision process with successive aircraft passes corresponding to stages. The basic factors to be considered at each stage are the weapon effectiveness, aircraft survival, and target acquisition and weapon delivery. ~~We determine~~ an optimal policy that indicates the number of weapons to be delivered and the mode of attack to be used at each pass depending on what state of affairs develops as the attack progresses. Various measures of effectiveness can be maximized subject to constraints on the number of passes that can be made, the number of weapons available to the aircraft, and the probability of the aircraft surviving the duel. The single aircraft attack models produce a "return-versus-attrition function" that is used in multiple aircraft raid models to determine the optimum raid size and the best policy for each aircraft. This determination minimizes the expected value of the number of aircraft lost in achieving a required level of return to the attackers. A multiple aircraft raid on multiple targets is considered where the problem is to allocate a given number of aircraft among targets and specify the policy for each aircraft to maximize the total utility to the attackers subject to suitable constraints. Uncertainty as to the true parameter values is approached by assuming complete ignorance of the value that each parameters might take within a specified range of uncertainty. A systematic method is developed that aids the decision maker to make a rational tactic selection considering that the parameter values might fall anywhere within their ranges of uncertainty.

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